Hypothetical extraction and fields of influence approaches: integration and policy implications*

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Abstract

We explicitly formulate (optimization) problems of finding a *key sector* and a *key group* of sectors within the framework of a hypothetical extraction method (HEM), and derive their solutions in terms of simple measures termed industries' *factor worths*. It is shown that the top $k \geq 2$ sectors with the largest total contributions to some factor, in general, do *not* constitute the key group of k sectors, the issue which is totally ignored in the input-output linkage literature. The link to the fields of influence approach is discovered, which gives an alternative economic interpretation for the HEM problems in terms of sectors' input self-dependencies. Further, we examine how a change in an input coefficient affects the importance of an industry. Also the link of the key sector analysis to sociology, network economics and coalitional game literature is explored. The key group problem is applied to the Australian economy for factors of water use, CO_2 emissions, and generation of profits and wages.

Keywords: key group of sectors, hypothetical extraction, fields of influence, social networks, coalitional game, redundancy, input-output

JEL Classification Codes: A14, D57, C67, E64, O21

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1 Introduction

There are ample studies within the input-output (IO) framework that investigate the issue of identification of so-called "key sectors" - sectors with the largest potential of spreading growth impulses throughout the economy. The issue of key sectors determination is seen to be useful for economic planning, in particular, in developing countries. From the development strategy point of view, it is reasonable for a country with a limited amount of financial resources to invest in those few industries, which have the largest impact on the whole economy through their buying and selling linkages with all other production units.[1] This approach, pioneered by Rasmussen (1956) and Hirschman (1958), was followed by a vast number of theoretical and empirical studies, and still constitutes one of the main areas in the IO and regional economics (see e.g., Strassert, 1968; Yotopoulos and Nugent, 1973; Jones, 1976; Schultz, 1977; Cella, 1984; Hewings et al., 1989; Heimler, 1991; Dietzenbacher, 1992; Sonis et al., 1995; Dietzenbacher and van der Linden, 1997; Cai and Leung, 2004; Cardenete and Sancho, 2006; Midmore et al., 2006; Beynon and Munday, 2008; Magtibay-Ramos et al., 2008).

However, the meaning of key sectors for economic development is rather debatable, since economic growth is determined not only by the structure and strength of inter-sectoral linkages, but also by production constraints, final demand and employment structure, imports, institutional and policy settings, income distribution, and technical and human capital endowment. Therefore, the application of key sector determination goes beyond examining only production linkages. For example, Diamond (1975); Meller and Marfán (1981); Groenewold et al. (1987, 1993) and Kol (1991) analyze employment linkages for Turkey, Chile, Australia, and for Indonesia, South Korea, Mexico and Pakistan, respectively. Gould and Kulshreshtha (1986) examine the impacts of final demand changes on energy use for Saskatchewan economy employing linkage analysis. Since according to the classical development economics for developing countries economic growth is intrinsically linked to changes in the structure of production, many studies applied the notion of key sectors to the analysis of structural change (see e.g., Hewings et al., 1989; Sonis et al., 1995; Roberts, 1995). Given current concerns about the environmental problems, Lenzen (2003) focuses on economic structure of Australia in terms of resource use and pollutant emissions by identifying key sectors and linkages that have large environmental impacts in the form of resource depletion and ecosystem degradation. Similarly,

[1] It is also true that the overall economic growth depends on the sectoral growth rates, which are in turn dependent on the linkages between the sectors. Strong linkages provide a possibility of gaining competitive advantage for industries. For instance, if a sector successfully enters a foreign market, it will be easier for industries (firms) that have high linkages with this sector to gain access to the foreign market as well (Porter, 1990; Hoen, 2002).

Sánchez-Chóliz and Duarte (2003), extending Rasmussen-type linkages, identify the key sectors in generating water pollution in the Aragonese economy.

In this paper we focus on the linkage analysis based on a *hypothetical extraction method* (HEM), which have become increasingly popular (Miller and Lahr, 2001). Just to mention a few recent studies, the HEM has been applied in the analysis of water use (Duarte et al., 2004), for the key sectors identification (Andreosso-O'Callaghan and Yue, 2004), in the analysis of the economy-wide roles of separate sectors, such as agriculture sector (Cai and Leung, 2004), construction sector (Song et al., 2006) and the real estate sector (Song and Liu, 2007). Los (2004) proposes to identify strategic industries using the HEM in a dynamic IO growth model. The HEM is also a useful tool to evaluate the significance of a sector in cases of crises-driven threats of industry shutdowns, which may help governments to decide whether to support financially the sector under threat or not.[2] The main contribution of this paper to the literature on key sectors identification from the HEM perspective is that we *explicitly* formulate the *optimization problems* of finding a *key sector* and a *key group of sectors*, and derive their solutions in terms of simple measures called industries' *factor worths.* The term "factor" refers to any indicator that interests an analyst in identifying the most important industries, which might be a social factor, such as employment, income, government revenue; or an environmental factor, such as primary energy consumption, greenhouse gas emissions, water use, land disturbance; or an economic/financial factor, such as GDP, gross operating surplus, export/import propensity; or any combinations of these factors. The important implications of our formal formulation of the HEM are the following.

Firstly, given that we have found simple measures for quantifying industries' importance, an analyst does *not* have to perform a three-step procedure of the HEM (to be explained in Section 3.1), which becomes, in particular, a rather formidable task when the number of industries is rather large (say, 100 or more). Secondly, and more importantly, we distinguish between a *key sector problem* and a *key group problem* and show that the key group of $k \geq 2$ sectors is, in general, *different* from the set of top k sectors selected on the base of the key sector problem. This is important, since up to date, to our best knowledge, the linkage literature accepted the top k sectors from the ranking of individual sectors contributions to the economy-wide output as the key group. This incongruence is due to the fact that while the key sector problem looks for the effect of extraction of one sector, the key group problem considers the effect of a *simultaneous* extraction of $k \geq 2$ sectors that takes

[2]The threat of downfall of the US car industry in the current financial crisis and debates on providing massive public spending to the industry can serve one such example. Other examples, are the downfall of the only Dutch aircraft manufacturer Fokker in 1995-96, and the disappearance of the Belgian national airline Sabena in 2001, both of which resulted in the shutdown of an entire national industry (Los, 2004).

differently into account the cross-contributions of the extracted industries to total factor arising within and outside the group. This impact is largely dependent on the similarity/dissimilarity of the linkage patterns of sectors to each other and of their final demand and factor generation structures. Thirdly, we show that the HEM is directly related to the fields of influence approach (Sonis and Hewings, 1989, 1992), which gives an alternative economic interpretation of the HEM problems in terms of the overall impact on aggregate factor due to an incremental change in sectors' input self-dependencies. Fourthly, our formulation of the HEM allows to examine a *combined* key sector/group problem, where the objective is a combination of several factors. For instance, one may wish to identify a key sector that has simultaneously the largest total (direct and indirect) contribution to economy-wide employment *and* the least total impact on carbon emissions generation.

Next, we examine the effect of a change in an input coefficient on the factor importance of an industry. It is shown that a positive (negative) change in the direct input coefficient a_{rc} never decreases (increases) the factor generating importance of any sector i, and surely increases (decreases) its factor worth if sector r requires directly and/or indirectly inputs from sector i. The economic interpretations of such change include, for example, an increase in complexity of technological links between sectors (or a rise in the density of the input matrix), an increase in sectoral interdependence, innovation and technological progress, etcetera.

Finally, we establish a bridge between the key linkage analysis in IO economics and the well established sociology literature on social networks and the booming literature of network economics on the one hand, and the coalitional game theory literature on the other. We explore on common grounds of the issues of finding a key sector in an economy and finding a key player in social networks. In particular, it is shown that the solution to these two tasks, defined, respectively, in terms of the *factor worth* and *intercentrality measure* are mathematically equivalent, where the last notion of network centrality was introduced by Ballester et al. (2006). It is also shown that there is a link to a class of solution concepts in coalitional game literature on *fair* allocation of gains obtained from cooperation among the coalition members, which also translates into defining the power of each member. In particular, we examine the connection between the industry factor worth and the Shapley value.

In our main empirical application of the key sector/group problem, we use the 1994-1995 Australian input-output tables and satellite accounts at 136 industry-level classification. We focus our analysis on two environmental, one financial, and one social factors, which are, respectively, water use, carbon dioxide (CO_2) emissions, gross operating surplus (profits), and wages and salaries. It is confirmed that the key group problem is not equivalent to the key sector problem. Most importantly, we find that Beef cattle and Electricity supply jointly generate 52.9% of Australian total (direct and indirect) CO_2 emissions, while Beef cattle, Dairy cattle, and Water supply, sewerage and drainage services jointly account for 48.1% of overall water consumption. Hence any attempt to reduce carbon emissions and more efficient use of water in Australia should target these industries in the first place.

The rest of the paper is organized as follows. Section 2 briefly discusses strengths and weaknesses of IO analysis, and presents some of its numerous current applications and extensions. In Section 3.1 we describe the planner's problem of finding a key sector, and examine how a change in a direct input coefficient affects the factor generating importance of industries. Section 3.2 generalizes the key sector problem to a key group identification problem, whose solution is defined in terms of a *group factor worth* of industries. The combined key sector/group problem is examined in Section 3.3. The link to the sociology, economics of social networks, and coalitional game theory is explored in Section 4. Section 5 contains results from the empirical application of the key sector and key group problems to the Australian economy. Section 6 concludes. In the Appendix we also provide results from application of the key sector/group problem to the Kyrgyzstan economy in 1997. All proofs are relegated to the Appendix.

2 Strengths and weaknesses of IO analysis

Any analytical model has its own strengths and weaknesses that are directly related to the validity of the model underlying assumptions. In what follows we first present the main limitations of Wassily Leontief's input-output (IO) model as it first appeared in the economics literature, and then briefly discuss how the IO economists and practitioners broadened its frontiers that in turn led to (complete) relaxation of some of these assumptions. Also we mention some of the many extensions and applications IO economics that are currently being used in addressing many kinds of economic analysis.

Traditionally IO model relied on the following limiting assumptions.

1. Each industry produces only one homogenous product.
2. Output is a linear function of final demand.
3. Technical coefficients are fixed, hence input substitution is not allowed.
4. Production in every industry is subject to constant returns to scale.
5. There are no input constraints, thus supply of inputs is perfectly elastic.
6. Prices and quantities are independent in the sense that IO quantity (price) model assumes fixed product prices (quantities).

Apparently these assumptions are unrealistic, hence quite restrictive. This also explains why IO analysis largely known to possess the above characteristics by non-

IO community is widely believed to be obsolete today. However, as surprising it may sound this field is still very much alive and its applications go far beyond those the method was originally designed for (see e.g., Miller and Blair, 2009).[3] Some of the numerous extensions and applications of IO analysis are the following:

- Commodity-by-industry accounting and models. This allows distinction between sectors and commodities, hence each sector may produce more than one product, and is the core of IO analysis today. Therefore, assumption 1 above needed for traditional IO studies is entirely relaxed.
- Interregional, multi-regional, and world IO models. These are designed to consider the complex interrelationships between regions within countries and/or between countries that are more prominent today due to globalization (see e.g., Duchin, 2005; Oosterhaven and Polenske, 2009).
- Social Accounting Matrices (SAM) as an extended IO model to capture activities of income distribution in the economy in a more comprehensive and integrated way.
- Structural Decomposition Analysis (SDA) to disaggregate the total amount of some factor into its various components such as, for example, technology change and final demand change.
- Dynamic IO models.
- Non-survey and partial survey methods, which are used in updating IO tables. This extension together with SDA and dynamic IO modeling imply that assumption 3 on fixed technical coefficients mentioned above is nowadays not as crucial as it was earlier, and is relaxed totally or partially (at least) in these studies.
- Energy and environmental IO analysis. This direction of IO framework usage is particularly quite popular nowadays, which is partly due to the fact that industrial ecologist, environmentalists and engineers seek to understand not only the direct effects but the full economy-wide impact of, say, alternative technologies for using energy and materials, and/or for generating damaging emissions (see e.g., the list of references in Foran et al., 2005, vol. 1, pp. 56-71).
- Nonlinear IO models to allow for variable economies of scale.
- Supply-side models, linkages and important coefficients (see e.g., Miller and Blair, 2009, Chapter 12 and references thereof).

[3]In the official website of the International Input-Output Association (http://www.iioa.org/) one can find the ongoing development of the field, which also provides some interesting links to institutions and projects around the globe that are doing IO research. See also *Eurostat Manual of Supply, Use and Input-Output Tables* (2008) at http://epp.eurostat.ec.europa.eu/ (ISBN: 978-92-79-04735-0).

- Multiplier decompositions, structural path analysis.
- Linear programming and optimization. This line of IO research considers explicitly resource constraints and treats price and quantity systems simultaneously, hence assumptions 5 and 6 above are no longer needed (see e.g., Ten Raa, 2005).
- Stochastic IO models and uncertainty calculus.
- IO econometric models, fundamental economic structure, qualitative IO analysis, variable IO models (see e.g., Miller and Blair, 2009, Chapter 14).

Also computable general equilibrium (CGE) models are build around IO coefficients table or SAM framework.[4] Ten Raa (2005) considers IO analysis "... probably the most practical tool of economic analysis" (p. xii), presents it from a mainstream economic perspective, and addresses a wide range of economic issues such as, for example, diagnosis of (in)efficiency of an economy, analysis of international trade, energy and environmental policy, productivity growth and spillovers, and derivation of Cobb-Douglas production function from IO coefficients. Dorfman (1995) believed that "any list of the four or five major advances in economics during the twentieth century will include input-output analysis" (p.305), while Baumol (2000) talking about the benefits of IO analysis states that it "provides a window to reality" and "permits applications that *really* can contribute to the well-being of society" (p. 151).

Evidently, one of the most important strengths of IO analysis is its ability to capture the extent of complex direct and indirect interconnectedness of sectors,[5] which explains why it is currently popular in other fields, for example, in the analysis of material flows, life cycle assessment, sustainable consumption, energy and climate change, waste management, etc. (see e.g., Suh, 2009). Also, IO model "is politically and ideologically neutral, and does not incorporate any specific behavioral conditions for the individual, companies or the state, except that an economy behave in a consistent manner" (Foran et al., 2005, p. 49).[6]

[4]Rose (1995) discusses the fact that many CGE modelers distance themselves from IO analysis, "including those standing on his [Leontief's] broad shoulders" and states: "My own use of CGE models had increased my appreciation on I-O economics rather than diminished it" (p. 296).

[5]In this regard Augusztinovics (1995) states: "Who knows what production structures and mathematical achievements will evolve in the future. Maybe two persons will be sufficient to supply energy and three to produce food for the whole of humanity on the surface of and around the globe, a few hundred thousand to teach youngsters via telecommunication and the rest will be busying themselves on various financial markets or cruising on spaceships in outer space. Even then, however, there will be a *division of labor* among them, separating their activities and at the same time tying them together through direct and endlessly circulating indirect links. ... Even if Mandelbrot sets will replace the square matrix, such models will be generalized successors of the original, the pioneering Input-Output" (p. 277).

[6]"For him [Leontief], economic theory and empirical research had to be closely linked ... . Amazingly, he was not deterred from thinking of working with the massive amounts of data that

3 Key sector problem vs. key group problem

3.1 Finding the key sector

The main point of departure is the open static Leontief model (see e.g., Miller and Blair, 2009), given by $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}$, where $\mathbf{x}$ is the $n \times 1$ endogenous vector of gross outputs of n sectors, $\mathbf{A}$ is the n-square input matrix representing technology, and $\mathbf{f}$ is the $n \times 1$ exogenous vector of final demands (including consumption, investments, and government expenditures).[7] The input coefficients a_{ij} denote the output in industry i directly required as input for one unit of output in industry j, hence the ith element of the vector $\mathbf{A}\mathbf{x}$ gives the total *intermediate* demand of all sectors for the output of industry i. That is, the fundamental equation of the open Leontief system states that gross output, $\mathbf{x}$, is the sum of all intermediate demand, $\mathbf{A}\mathbf{x}$, and final demand, $\mathbf{f}$. The reduced form of the model is

$$\mathbf{x} = \mathbf{B}\mathbf{f}, \tag{1}$$

where $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$ is the Leontief inverse with $\mathbf{I}$ being the identity matrix. Its element b_{ij} denotes the output in industry i directly and indirectly required to satisfy one unit of final demand in industry j. The row vector of *output multipliers* is defined as $\mathbf{m}_o' = \boldsymbol{\imath}'\mathbf{B}$, where $\boldsymbol{\imath}$ is a summation vector consisting of ones. Its j-th element $m_j^o = \sum_{k=1}^{n} b_{kj}$ indicates the increase of total output in all industries per unit increase of final demand in industry j.

For the purpose of identification of important sectors we adopt the *hypothetical extraction method* (HEM) originally developed and used by Paelinck et al. (1965), Strassert (1968) (as cited in Miller and Lahr, 2001) and Schultz (1977), the central idea of which is briefly as follows. To estimate the importance of sector i to the economy, delete the i-th row and column of the input matrix $\mathbf{A}$, and then using (1) compute the reduced outputs in this hypothetical case (the final demand vector also excludes f_i). The difference between *total* outputs of the economy before and after the extraction (called "total linkage") measures the relative stimulative importance of sector i to the economy.[8]

exist in the real world. I can only wonder why other great theorists shy away from such work." (Polenske, 2004, p. 25).

[7] Adopting usual convention, matrices are given in bold, uppercase letters; vectors in bold, lowercase letters; and scalars in italic lowercase letters. Vectors are columns by definition, and transposition is indicated by a prime.

[8] This method was criticized for the reason that it does not distinguish the total linkages into backward and forward linkages (see e.g., Meller and Marfán, 1981; Cella, 1984; Clements, 1990; Dietzenbacher and van der Linden, 1997). However, we believe that for measuring a sector's economy-wide impact it is the most adequate HEM, since setting to zero only a column (row) to compute the backward (forward) linkages in the non-complete HEM takes only one-sided impact

However, unlike the traditional HEM approach, we allow for a rather general definition of importance, which may be used to address various economic, social, and/or environmental issues.[9] For instance, key sectors may be determined according to their potential of generating income, emission of greenhouse gases, creating jobs, or resource use. For the purpose of a general exposition of the HEM problem, we refer to the various policy-relevant indicators as *factors*. Let the vector of *direct factor coefficients* $\boldsymbol{\pi}$ denotes the sectoral factor usage per unit of total output, hence the row vector of *factor multipliers* is $\mathbf{m}'_\pi = \boldsymbol{\pi}'\mathbf{B}$, and its j-th element $m^\pi_j = \sum_{k=1}^n \pi_k b_{kj}$ indicates the economy-wide increase of factor usage/production per unit increase of final demand in industry j.

We are now in a position to address the key sector identification problem. Let first denote by $\mathbf{A}^{-i}$ the new input matrix derived from $\mathbf{A}$ by setting to zero all of its i-th row and column elements. The crucial assumption made (which is usual for all the HEM approaches) is that in a new system without sector i the input structure of all sectors $j \neq i$ remains unchanged. From economic point of view, this implies that foreign (external) industries substitute sector i in providing its output in order to satisfy the intermediate demand of the remaining industries and the final demand for commodity i. Although at first glance this assumption seems restrictive, in fact it is not given our main aim of identifying the importance of sector i. The point is that by taking all other input coefficients fixed, we explicitly allow the resulting outcome to depend only on extraction of sector i, which is now not participating in the "roundabout" of the production process. The vector of total outputs after extracting sector i is $\mathbf{x}^{-i} = \mathbf{B}^{-i}\mathbf{f}^{-i}$, where $\mathbf{B}^{-i} = (\mathbf{I} - \mathbf{A}^{-i})^{-1}$, and $\mathbf{f}^{-i}$ is the same as $\mathbf{f}$ except its i-th entry that is set to zero. The reason for excluding f_i in the final demand vector $\mathbf{f}^{-i}$ is that when sector i ceases to exist, its (domestic) output should be zero, which from (1) is equivalent to $f_i = 0$ (see also e.g., Schultz, 1977; Miller and Lahr, 2001).

The objective is picking the appropriate sector i, such that its extraction from the system generates the highest possible reduction in the factor of interest (say, total income). Formally, the problem is

$$\max\{\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-i} \,|\, i = 1, \ldots, n\}. \tag{2}$$

into account. Moreover, the last two linkage measures are closely related in the sense of the forward-link involvement problem of backward linkage measures, and, vice versa, the backward-link presence in the forward linkage measures (see e.g., Yotopoulos and Nugent, 1973; Cai and Leung, 2004). See Miller and Lahr (2001) for an excellent discussion on all possible extractions, who state that for the purpose of finding a key sector "we believe the original hypothetical extraction approach ... is totally adequate - Meller and Marfán and other modifications notwithstanding" (p. 429).

[9]For example, Ten Raa (2005, p. 26) states: "Output increases induced by a final demand stimulus are of little interest in themselves. What matters is the income generated by the additional economic activity."

This is a finite optimization problem, which is solvable. A solution to (2) is denoted by i^* and is called the *key sector*. Removing i^* from the initial production structure has the largest overall impact on the factor generation. Before solving (2), let first briefly introduce Sonis and Hewings' notion of a *field of influence*, which is another technique for evaluating a sector's influence on the rest of the economy (see e.g., Sonis and Hewings, 1989, 1992).[10] Let consider a change of $\alpha \neq 0$ in only one coefficient a_{rc}, with all other input coefficients being fixed. Then the Leontief inverse after the change is[11]

$$\widetilde{\mathbf{B}} = \mathbf{B} + \frac{\alpha}{1 - \alpha b_{cr}} \mathbf{F}(r, c), \tag{3}$$

where the *first-order field of influence* of the coefficient a_{rc} is $\mathbf{F}(r, c) = \mathbf{B}\mathbf{e}_r\mathbf{e}_c'\mathbf{B}$ and $\mathbf{e}_r$ is the r-th column of the identity matrix. The sum of all elements of the first-order field of influence matrix, $\boldsymbol{\imath}'\mathbf{F}(r, c)\boldsymbol{\imath}$, gives the *first-order intensity field of influence* of the direct input a_{rc}. In Sonis and Hewings (1989) this concept was introduced in order to measure the *inverse importance* of direct inputs. Consequently, those elements of $\mathbf{A}$ whose changes lead to the largest impact on the system are called the *inverse-important coefficients*.

To solve (2) we use the following result due to Zeng (2001, Theorem 1, p. 304), our proof of which is given in the Appendix.[12]

Lemma 1. *Let $\mathbf{B}$ and $\mathbf{B}^{-i}$ be, respectively, the Leontief inverses before and after extraction of sector i from the production system, and $\mathbf{e}_i$ be the i-th column of the identity matrix. Then $\mathbf{B} - \mathbf{B}^{-i} = \frac{1}{b_{ii}}\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\mathbf{B} - \mathbf{e}_i\mathbf{e}_i'$.*

Using Lemma 1 after some mathematical transformations, the planner's problem (2) can be rewritten as (see Appendix):

$$\max \left\{ \frac{1}{b_{ii}} \boldsymbol{\pi}'\mathbf{F}(i, i)\mathbf{f} \;\middle|\; i = 1, \ldots, n \right\}. \tag{4}$$

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[10] This methodology answers the question of how changes in some elements of the input matrix affect the rest of the system by examining the impact on the elements of the Leontief inverse, and is general enough to handle changes in one direct coefficient, in all elements of a row or column of the input matrix, or in all coefficients simultaneously. From economic point of view this enables one to analyze the effect of technological change, improvements in efficiency, changes in product lines, changes in the structure and complexity of an economy over time, changes in trade dependency of a country, etc.

[11] Notice that $\frac{\partial \tilde{b}_{ij}}{\partial \alpha}\big|_{\alpha=0} = f_{ij}(r, c) = b_{ir}b_{cj} = f_{cr}(j, i)$. Also the coordinate form of (3) is the well-know Sherman and Morrison (1950) formula of inverse change as $\tilde{b}_{ij} = b_{ij} + \alpha b_{ir} b_{cj}/(1 - \alpha b_{cr})$.

[12] Independently, also Ballester et al. (2006, Lemma 1, p. 1411) establish the same result in a social network framework. We should note that their Lemma 1 is given for a *symmetric* adjacency matrix , and does not consider the ii-th element of the difference $\mathbf{B} - \mathbf{B}^{-i}$. For asymmetric case, change $m_{ij}(\mathbf{g}, a)$ to $m_{ji}(\mathbf{g}, a)$ in their Lemma 1. Although Ballester et al. (2006) investigate identification of a key player in social networks, there is a direct link to the key sector problem, which will be discussed in Section 4.1.

Re-expression of (2) in terms of the problem (4) clearly shows that the fields of influence approach and the complete HEM are closely related. This is not surprising since both deal with the same issue of the impact of a change in input coefficients. The effect of a change in direct self-dependency of sector i, i.e., a change in the input coefficient a_{ii}, on the Leontief inverse is given by the first-order field of influence $\mathbf{F}(i,i)$. To give more economic interpretation to the problem (4), we introduce the following terms. In contrast to the standard first-order intensity field of influence of the input coefficient a_{rc}, $\boldsymbol{\imath}'\mathbf{F}(r,c)\boldsymbol{\imath}$, the scalar $\boldsymbol{\imath}'\mathbf{F}(r,c)\mathbf{f}$ is termed the *output first-order intensity weighted field of influence* of a_{rc}, which weights every purchasing sector in the sum according to the size of its final demand. This makes more sense in computing the global intensity since every sector is not given an equal importance, but rather its scale of final demand satisfaction is taken into account. Similarly, we term the scalar $\boldsymbol{\pi}'\mathbf{F}(r,c)\mathbf{f}$ as the *factor first-order intensity weighted field of influence* of the coefficient a_{rc}, since the last measures the effect of an input change on total factor generation rather than gross output. Hence, the planner's problem (4) searches for such sector i that, on the one hand, has a large economy-wide impact on total factor usage/generation due to (incremental) change in its *direct input self-dependency*, and on the other hand, is less input dependent on itself directly and indirectly.

The problem in (2) is equivalent to $\min\{\boldsymbol{\pi}'\mathbf{x}^{-i}\,|i=1,\ldots,n\}$. However, a direct use of one of these criteria in determining the key sector in empirical applications forces an analyst to extract different sectors separately and compute the required objective n times, which becomes a formidable task when n is large. Although with the modern technology this is not a big issue, problem (4) shows that there exist a much simpler way to get the desired outcome. We define the *factor worth* of sector i as

$$\omega_i^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})=\frac{m_i^{\pi}x_i}{b_{ii}}.$$

Thus, given the planner's objective (4) and the fact that $\boldsymbol{\pi}'\mathbf{F}(i,i)\mathbf{f}=\boldsymbol{\pi}'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{f}=m_i^{\pi}x_i$, we have established the following result.

Theorem 1. *The key sector i^* that solves* $\max\{\boldsymbol{\pi}'\mathbf{x}-\boldsymbol{\pi}'\mathbf{x}^{-i}\,|\,i=1,\ldots,n\}$ *has the highest factor worth, i.e.,* $\omega_{i^*}^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})\geq\omega_i^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi})$ *for all* $i=1,\ldots,n$.

From Theorem 1 it follows that the standard measure of the high factor multiplier m_i^{π} is not sufficient for sector i to be an optimal target, say, for investments. For the last, besides m_i^{π}, the size of the sector's output x_i and its total input self-dependency as indicated by b_{ii} are equally important, where the first has a positive effect, while the second an inverse effect on the worth of sector i.

The traditional gross output approach of the HEM corresponds to the problem (2) or (4) when a summation vector $\boldsymbol{\imath}$ is substituted for the vector of factor coefficients $\boldsymbol{\pi}$. The following result is then an immediate outcome of Theorem 1.

Corollary 1. *The key sector i^* that solves* $\max\{\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}^{-i} \mid i = 1, \ldots, n\}$ *has the highest gross output worth, i.e., $\omega^o_{i^*}(\mathbf{A}, \mathbf{f}) \geq \omega^o_i(\mathbf{A}, \mathbf{f})$ for all $i = 1, \ldots, n$, where $\omega^o_i(\mathbf{A}, \mathbf{f}) = m^o_i x_i / b_{ii}$ is the gross output worth of sector i.*

Notice that the gross output factor worth of sector i is nothing else as the "total linkage" of a sector as defined in the classical HEM approach.

Next we examine how stronger interdependence of sectors affect the factor worth of sector i. Let the input matrix $\widetilde{\mathbf{A}}$ represent the more "complex" input structure than $\mathbf{A}$, and, without loss of generality, assume that $\widetilde{\mathbf{A}}$ differs from $\mathbf{A}$ only with respect to the rc-th element that is increased by $\alpha > 0$. Then it is apparent that $\widetilde{\mathbf{B}} = \mathbf{I} + \widetilde{\mathbf{A}} + \widetilde{\mathbf{A}}^2 + \cdots > \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots = \mathbf{B}$,[13] which in turn implies that, given $\mathbf{f}$ and $\boldsymbol{\pi}$, both the numerator and denominator in the definition of the factor worth of sector i might only increase, hence it is not clear whether $\omega^\pi_i(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi})$ is larger or smaller than $\omega^\pi_i(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$. Nevertheless, in Theorem 2 below we are able to show that a rise in direct input interdependence between two sectors never decreases sector i's factor worth, and, moreover, we establish a necessary and sufficient condition under which such a change surely increases $\omega^\pi_i(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi})$.

Theorem 2. *Let the input matrix $\widetilde{\mathbf{A}}$ differ from $\mathbf{A}$ only with respect to the rc-th entry, which has been changed by $\alpha \neq 0$. Given a nonnegative $\mathbf{f}$ with $f_c > 0$ and $\boldsymbol{\pi}$, if $\alpha \gtrless 0$ then*

(i) $\omega^\pi_i(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi}) \gtrless \omega^\pi_i(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ for $i = r, c$;

(ii) $\omega^\pi_i(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi}) \gtreqqless \omega^\pi_i(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ for all $i \neq r, c$, with equality holding if and only if $b_{ir} = b_{ci} = 0$.

One implication of Theorem 2 is that when domestic industries become more interdependent on each other, then the factor generating importance falls for no sector and surely increases for sectors directly involved in this higher input dependencies. Moreover, any other sector i's worth increases as well if $b_{ir} > 0$ and/or $b_{ci} > 0$. The second implication is that a more efficient technology never increases the factor worth of any sector for the same vectors of final demand and factor coefficients. In particular, if, say, due to innovation a_{rc} decreases, then sectors r and c's factor worths strictly decrease and any other sector i's importance also weakens whenever $b_{ir} > 0$ and/or $b_{ci} > 0$. These two conditions in both cases imply that sector i should either provide (directly and/or indirectly) inputs to industry r and/or uses inputs (directly and/or indirectly) from sector c.

The straightforward special case of Theorem 2 is when $\boldsymbol{\pi} = \boldsymbol{\imath}$, which shows that the gross output worth of sector i increases (decreases) if the input coefficient a_{rc}

[13] We write $\mathbf{X} > \mathbf{Y}$ if $x_{ij} \geq y_{ij}$ for all i, j, with at least one strict inequality.

increases (decreases) and sector r purchases inputs directly and/or indirectly from industry i.

Corollary 2. *Assume that the input coefficient a_{rc} changes by $\alpha \neq 0$, i.e., $\tilde{a}_{rc} = a_{rc} + \alpha$. Then, given a nonnegative $\mathbf{f}$ with $f_c > 0$, $\omega_i^o(\widetilde{\mathbf{A}}, \mathbf{f}) \gtrless \omega_i^o(\mathbf{A}, \mathbf{f})$ for $i = r, c$, and $\omega_i^o(\widetilde{\mathbf{A}}, \mathbf{f}) \gtreqqless \omega_i^o(\mathbf{A}, \mathbf{f})$ for all $i \neq r, c$ whenever $\alpha \gtrless 0$, with equality holding if and only if $b_{ir} = b_{ci} = 0$.*

3.2 Finding the key group

Although the linkage literature using the HEM acknowledges the possibility of extraction of several industries, the theoretical analysis does not go beyond describing it using partitioned matrices to the reduced form of the Leontief model (see e.g., Miller and Lahr, 2001). This, however, is quite complex to implement empirically since one has to consider all possible combinations of certain number of industries from totality of n sectors (correspondingly changing the members and nonmembers of partitioned matrices) in order to determine the most important group of sectors, which explains why there is no any empirical study that explicitly focuses on the role of several industries simultaneously. Hence, in all studies, to the best of our knowledge, the HEM was applied to only one sector, and the most important industries were defined to be those with the largest individual contributions to total output (or any other factor).

In this section we wish to fill this gap in the literature, generalizing the key sector problem from the previous section to the *key group problem.* Similar to the notion of individual key sector, a *key group* of $k \geq 2$ sectors is defined as the group of industries, whose removal from the production system has the largest impact on the factor consumption/generation.[14] Since the two problems are inherently different, we expect that, in general, the top k sectors with the largest factor worths do *not* compose the *key group*, which is also confirmed in the empirical application in Section 5. The underlying reason for this outcome is that industries can be *redundant* (or, equivalently, similar to each other) with respect to their linkage patterns to other sectors and their capabilities of factor generation. Hence, all other things being equal, targeting industries with the same linkage characteristics might not be an optimal policy strategy and instead choosing sectors with *heterogenous* linkage structures will have the largest impact on the factor usage/generation.

The planner's objective is now to pick k ($1 \leq k \leq n$) sectors $i_1, i_2, \dots, i_k$ ($i_s \neq i_r$)

[14] Note that if the factor generation is unfavorable from societal point of view (e.g., an increase in CO_2 emissions has detrimental consequences) and the policy-makers want to find the *least* harmful industries to target on, then the key group will be defined as those industries that have the *smallest* impact on the factor generation.

such that their extraction from the production structure generates the largest impact on the overall factor consumption/generation, i.e.,

$$\max\{\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-\{i_1,\dots,i_k\}} \,|i_1,\dots,i_k = 1,\dots,n; i_s \neq i_r\}, \tag{5}$$

where $\mathbf{x}^{-\{i_1,\dots,i_k\}} = \mathbf{B}^{-\{i_1,\dots,i_k\}}\mathbf{f}^{-\{i_1,\dots,i_k\}}$, and the superscript $-\{i_1,\dots,i_k\}$ refers to the situation when sectors $i_1, i_2, \dots, i_k$ are hypothetically extracted from the economy. Note that along the similar reasonings made in Section 3.1, $\mathbf{f}^{-\{i_1,\dots,i_k\}}$ is exactly the same as $\mathbf{f}$ but with $f_{i_s} = 0$ for all $s = 1,\dots,k$. The solution to (5) is denoted by $\{i_1^*, i_2^*, \dots, i_k^*\}$ and is called the *key group of size k*.

The following important identity characterizes the changes in all elements of the Leontief inverse when a group of k sectors is hypothetically extracted from the production system.

Lemma 2. *Let $\mathbf{B}^{-\{i_1,\dots,i_k\}}$ be the Leontief inverse after extraction of sectors $i_1, i_2, \dots, i_k$ from the production system, where $1 \leq k \leq n$, and $\mathbf{e}_i$ be the i-th column of the identity matrix. Then the identity $\mathbf{B} - \mathbf{B}^{-\{i_1,\dots,i_k\}} = \mathbf{BE}\left(\mathbf{E}'\mathbf{BE}\right)^{-1}\mathbf{E}'\mathbf{B} - \mathbf{EE}'$ always holds, where $\mathbf{E}$ is the $n \times k$ matrix defined as $\mathbf{E} = (\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k})$.*

Note that Lemma 1 is just a special case of Lemma 2 with $k = 1$. We should also note that the k extracted sectors can be arbitrary ordered, hence the matrix $\mathbf{E}$ can have different ordering of the identity columns corresponding to those sectors.[15]

Using Lemma 2 it can be shown that the problem in (5) is exactly equivalent to (see Appendix)

$$\max\left\{\boldsymbol{\pi}'\mathbf{BEB}_{kk}^{-1}\mathbf{E}'\mathbf{Bf} = \frac{\boldsymbol{\pi}'\mathbf{F}[(i_1,i_1),\dots,(i_k,i_k)]\mathbf{f}}{|\mathbf{B}_{kk}|}\,\middle|\, i_1,\dots,i_k = 1,\dots,n; i_s \neq i_r\right\}, \tag{6}$$

where $\mathbf{B}_{kk} = \mathbf{E}'\mathbf{BE}$, $|\mathbf{B}_{kk}|$ is the determinant of $\mathbf{B}_{kk}$, and $\mathbf{F}[(i_1,i_1),\dots,(i_k,i_k)]$ is the matrix field of influence of order k of the coefficients $a_{i_1i_1}, a_{i_2i_2}, \dots, a_{i_ki_k}$.

Note that the key player problem (4) is a particular case of (6) when $k = 1$. So the overall impact on the Leontief inverse of a simultaneous incremental change in coefficients $a_{i_1i_1}, \dots, a_{i_ki_k}$ is given by the matrix $\mathbf{F}[(i_1,i_1),\dots,(i_k,i_k)]$, and thus the scalar $\boldsymbol{\imath}'\mathbf{F}[(i_1,i_1),\dots,(i_k,i_k)]\boldsymbol{\imath}$ is the standard *intensity field of influence of order* k of the corresponding input coefficients. Due to similar reasonings made in Section 3.1, we term the scalar $\boldsymbol{\imath}'\mathbf{F}[(i_1,i_1),\dots,(i_k,i_k)]\mathbf{f}$ as the *output intensity weighted field of influence of order* k of the input coefficients $a_{i_1i_1}, \dots, a_{i_ki_k}$, which unlike the standard global intensity weights each purchasing sector according to its net output. Similarly,

[15]Notice that if $k = n$ and $\mathbf{E} = \mathbf{I}$, then $\mathbf{B} - \mathbf{B}^{-\{i_1,\dots,i_k\}} = \mathbf{B} - \mathbf{I}$, which is expected. However, in this case also $\mathbf{E}$ does not have to be an identity matrix, but $\mathbf{E}$ being any permutation matrix of order n gives the desired result.

$\boldsymbol{\pi}'\mathbf{F}[(i_1, i_1), \ldots, (i_k, i_k)]\mathbf{f}$ can be termed as *factor intensity weighted field of influence of order* k of $a_{i_1 i_1}, \ldots, a_{i_k i_k}$ that measures the effect of changes in these coefficients on factor generation/usage.[16] Now define the *group factor worth* of sectors $i_1, \ldots, i_k$ ($i_r \neq i_s$) as

$$\omega^{\pi}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \mathbf{m}'_{\pi}\mathbf{E}\mathbf{B}^{-1}_{kk}\mathbf{E}'\mathbf{x}.$$

The following result then follows from the planner's problem (6).

Theorem 3. *For* $1 \leq k \leq n$ *the key group of size* k $\{i_1^*, i_2^*, \ldots, i_k^*\}$ *that solves* $\max\{\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-\{i_1,\ldots,i_k\}} \,|i_1, \ldots, i_k = 1, \ldots, n; i_s \neq i_r\}$ *has the highest group factor worth, i.e.,* $\omega^{\pi}_{i_1^*,\ldots,i_k^*}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) \geq \omega^{\pi}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ *for all* $i_1, \ldots, i_k = 1, \ldots, n$ *with* $i_s \neq i_r$*.*

Hence, in line with the key sector problem, the planner's problem (5) searches for a group of k sectors with the highest k-th order group factor worth, which is directly proportional to the impact on overall factor generation of an incremental changes in *direct input self-dependencies* of the sectors comprising the group, and inversely related to their unit *own net input dependence* that excludes the indirect role of the group members. To see the interpretation of the second effect, let consider the group of size two. Then $|\mathbf{B}_{22}| = b_{ii}b_{jj} - b_{ij}b_{ji}$, which gives the *net* input dependence per unit of output of sectors i and j ($\neq i$) on themselves. This follows since $b_{ij}b_{ji}$ (or, equivalently, $b_{ji}b_{ij}$) gives the total input requirements of sector i (j) on itself through sector j (i), and excluding this from the total own dependence of sectors i and j, $b_{ii}b_{jj}$, gives the unit own input dependence through other sectors $k \neq i, j$.[17]

When the key group of size k is searched in the spirit of the traditional HEM approach, Theorem 3 implies the following result.

Corollary 3. *For* $1 \leq k \leq n$ *the key group of size* k $\{i_1^*, \ldots, i_k^*\}$ *that solves* $\max\{\boldsymbol{\imath}'\mathbf{x} - \boldsymbol{\imath}'\mathbf{x}^{-\{i_1,\ldots,i_k\}} \mid i_1, \ldots, i_k = 1, \ldots, n; i_s \neq i_r\}$ *has the highest group output worth, i.e.,* $\omega^{o}_{i_1^*,\ldots,i_k^*}(\mathbf{A}, \mathbf{f}) \geq \omega^{o}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f})$ *for all* $i_1, \ldots, i_k = 1, \ldots, n$ *with* $i_s \neq i_r$ *and* $\omega^{o}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f}) = \mathbf{m}'_{o}\mathbf{E}\mathbf{B}^{-1}_{kk}\mathbf{E}'\mathbf{x}$*.*

While the key sector problem looks for the effect of extraction of one sector, the key group problem considers the effect of a *simultaneous* extraction of $k \geq 2$ sectors. This implies that the two problems are not equivalent since the key group problem takes into full account all the cross-contributions of the extracted

[16] Using the analytical formula for matrix inverse it is easy to show that, for example, the factor second-order intensity weighted field of influence of sectors i and j ($\neq i$) is $\boldsymbol{\pi}'\mathbf{F}[(i,i),(j,j)]\mathbf{f} = b_{jj}m^{\pi}_i x_i + b_{ii}m^{\pi}_j x_j - b_{ij}m^{\pi}_i x_j - b_{ji}m^{\pi}_j x_i$.

[17] In case of three sectors, one may write $|\mathbf{B}_{33}| = b_{kk}(b_{ii}b_{jj} - b_{ij}b_{ji}) - b_{jk}(b_{ii}b_{kj} - b_{ki}b_{ij}) - b_{ik}(b_{jj}b_{ki} - b_{kj}b_{ji})$ for all $i \neq j \neq k \neq i$ which has the same interpretation of the net own input dependence of sector k. Other orderings of rows (and columns) of $\mathbf{B}_{kk}$ give similar interpretation for the other two sectors i and j.

sectors to the overall factor that is used/generated both within and outside the group. For example, if two industries are perfectly identical with respect to their linkages patterns (including input coefficients' sizes) and more or less also similar in terms of their final demand and factor generation structure, then their group worth is expected to be less than that of the group, which consists of one of the mentioned sectors together with another industry that has quite different patterns of (significant) interindustry linkages and factor generation ability. The *redundancy principle* is well-known in the sociology literature on social networks that emphasizes the redundancy of actors with respect to adjacency, distance, and bridging (see e.g., Burt, 1992; Borgatti, 2006). Arguing that the information and control benefits of a large and *diverse* network are more than those of a small and homogeneous network, Burt (1992, p.17), for example, states: "What matters is the number of nonredundant contacts. Contacts are redundant to the extent that they lead to the same people, and so provide the same information benefits." Taking redundancy into account is crucial in determining the most important group in social networks (see Everett and Borgatti, 1999, 2005; Temurshoev, 2008). In the IO framework, however, it is not only the redundancy of sectors with respect to their production linkage that matters, but also the similarity of the structures of sectors final demands and factor production is important in determining the key group.

In general, within the IO framework, we expect that k (≥ 2) sectors with the largest individual factor worths will not be much different from the key group of size k only if the IO tables are highly aggregated. Otherwise, the difference should be in place, and will largely depend on the structures of the production system, direct factor coefficients and final demands.

3.3 The combined key sector/group problem

Nowadays policy-makers, governments, companies and the general public are all becoming engaged with the phenomenon of "sustainability", which was brought to the public attention by environmental movements about 30 years ago that mainly emphasizes the issue of some sort of tradeoff between economic development and environmental quality. Hence, the concept of sustainable development is becoming the main focus, which requires meeting increasing environmental concerns along with maintaining economic development. For this reason corporations are beginning to be more and more involved in using the so-called *triple bottom line* (TBL) accounting through which economic, social and environmental spheres of sustainability are assessed and reported (see e.g., Henriques and Richardson, 2004). Further, at the country level Foran et al. (2005) develop a numerate TBL account of the Australian economy with ten indicators that accounts for the full supply chain approach using

IO analysis, against which many management issues at lower (say, firms) levels can be benchmarked.

The generalized HEM approach proposed in this paper can be applied to the sustainable development policy design and analysis.[18] The key group problem (5) can easily accommodate the notion of TBL approach from the HEM perspective. Let take the economic, social and environmental factors in the example of value-added, employment and CO_2 emissions, respectively. If $\mathbf{v}$, $\mathbf{l}$ and $\mathbf{c}$ denote, respectively, the direct value-added, labor and CO_2 coefficients vectors, the total (direct and indirect) value-added, employment and CO_2 emissions that is generated to satisfy the final demand $\mathbf{f}$ is equal to $\mathbf{v}'\mathbf{x}$, $\mathbf{l}'\mathbf{x}$ and $\mathbf{c}'\mathbf{x}$, correspondingly. Then a *combined key sector* and a *combined key group* problems are given, respectively, by problems (2) and (5) with the direct factor coefficients defined as $\boldsymbol{\pi} = \mathbf{v} + \mathbf{l} - \mathbf{c}$. Note that since CO_2 generation is unfavorable, its direct coefficients are entered with a minus sign in the definition of $\boldsymbol{\pi}$. Also notice that factors written is this form can have an economic meaning only if they are all expressed in the same measurement unit. This can be done, for example, by multiplying the number of jobs by a price so that employment is expressed in some common for all factors currency term (like in the index number literature). Or, one might assign appropriate weights to each factor that is included in the combined factor coefficients vector. For instance, we may write $\mathbf{l} = t_v\mathbf{j}$, where the (number of) jobs coefficients $\mathbf{j}$ is expressed in terms of currency using the weight $t_v = \frac{\mathbf{v}'\mathbf{x}}{\mathbf{j}'\mathbf{x}}$ that indicates the value of value-added per one (full-time) job. Theorems 1 and 3 are then similarly used to identify the key sector and the key group of certain size in these combined problems.

4 Connection to social network analysis and game theory

4.1 The link to the economics of social networks

One of the topical issue in sociology literature is the problem of identifying a *key player* in a social network. Different measures of network centralities were proposed

[18] An example of such policy is given by Daniels (1992): since the 1980s Australia has expanded its exports of meat, wool, wheat and non-ferrous metals to maintain revenues and living standards in response to increasing foreign debts and falling primary commodity prices. However, since these exports are highly environmental damaging activities, "Australia became locked into an environmental-economic dilemma through increasing dependency on degrading production and further erosion of environmental quality. Daniels argued that, in order to avoid long-term losses of productivity, biodiversity and real income, Australia has to re-direct its domestic production towards more value-adding and less land- and emissions-intensive commodities" (Lenzen, 2003, p. 29).

for this purpose, such as rank prestige, centralities of degree, closeness, betweenness and information (see e.g., Katz, 1953; Sabidussi, 1966; Freeman, 1979; Bonacich, 1987; Stephenson and Zelen, 1989).[19] From an economic point of view, the important feature of network games is that actors' payoffs depend on each other through the network embeddedness.[20] A set of players each choose a level of some activity in a game, where there are negative global externalities (e.g., competition) and local positive externalities (e.g., learning, cooperation) that come through the network. This system has feedback effects, which are taken into account in the Nash equilibrium activity levels. Recently, this network game was analyzed by Ballester et al. (2006, henceforth BCZ), who show that its *individual* equilibrium level is proportional to the Katz-Bonacich centrality measure from the sociology literature, hence provide to the Katz and Bonacich indices behavioral foundation "singling [them] out from the vast catalogue of network measures" (p. 1404). They also propose a new measure, named *intercentrality measure*, that finds a key player with the maximum influence on overall activity. In what follows, we briefly present a simple version of BCZ (2006) model, and discuss the link between the intercentrality measure and the factor worth proposed in this paper.

Players are connected by a network $\mathbf{g}$ with adjacency matrix $\mathbf{G}$, which is symmetric, zero-diagonal, and non-negative square matrix with the typical element $g_{ij} = \{0, 1\}$ for all $i \neq j$. If players i and j ($\neq i$) are connected, then $g_{ij} = 1$, otherwise $g_{ij} = 0$. Each player $i = 1, \ldots, n$ selects a contribution $y_i \geq 0$ and gets the bilinear payoff

$$u_i(\mathbf{y}, \mathbf{g}) = y_i - \frac{1}{2}y_i^2 + a\sum_{j=1}^{n} g_{ij}y_iy_j, \tag{7}$$

which is strictly concave in own contribution, $\partial^2 u_i/\partial y_i^2 = -1 < 0$, hence marginal utility of player i is decreasing in own activity level. We set $a > 0$ to capture the network payoff (relative) complementarities across all pairs of actors, which are reflected by the cross-derivatives $\partial^2 u_i/\partial y_i \partial y_j = a g_{ij} \geq 0$ for $i \neq j$. That is, marginal utility of actor i is increasing in actor j's contribution if $g_{ij} = 1$, otherwise there is no direct effect.

Denote the largest eigenvalue of $\mathbf{G}$ by $\mu(\mathbf{G}) > 0$. Then if $a\mu(\mathbf{G}) < 1$, the matrix $\mathbf{M}(\mathbf{g}, a) = (\mathbf{I} - a\mathbf{G})^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k$ is well defined,[21] and its coefficients $m_{ij}(\mathbf{g}, a)$ count the number of paths in $\mathbf{g}$ starting at i and ending at j, where paths of length k are weighted by a^k. Hence, the parameter a is a decay factor that scales down the

[19] A thorough discussion of centrality and many more reference can be found in Wasserman and Faust (1994, pp. 169-219).

[20] Economists mainly use modern game theoretical tools in analyzing social networks (see e.g., Goyal, 2007; Jackson, 2008).

[21] This follows from Theorem III* in Debreu and Herstein (1953, p. 601).

weight of longer paths. The vector of Katz-Bonacich (KB) centralities of parameter a in $\mathbf{g}$ is $\mathbf{h}(\mathbf{g}, a) = \mathbf{M}(\mathbf{g}, a)\boldsymbol{\imath}$, and its i-th component $h_i(\mathbf{g}, a) = \sum_{j=1}^{n} m_{ij}(\mathbf{g}, a)$ indicates the *total number of direct and indirect paths* in $\mathbf{g}$ that start from position i.[22] Note that, similar to the Leontief inverse property, $m_{ii}(\mathbf{g}, a) \geq 1$, hence $h_i(\mathbf{g}, a) \geq 1$ with equality holding when i is an isolate.

From Theorem 1 in BCZ (2006) it is easy to see that for $a\mu(\mathbf{G}) < 1$, the unique Nash equilibrium of the game is $\mathbf{y}^* = \mathbf{h}(\mathbf{g}, a)$. One can also easily derive that the equilibrium utility of player i is $u_i(\mathbf{y}^*, \mathbf{g}) = h_i(\mathbf{g}, a)^2/2$. This shows that individual equilibrium outcomes are directly related to the KB centrality measures. Next, let consider a *key player* in the framework of social networks. The key player i^* is a solution to the planner's problem $\max\{\boldsymbol{\imath}'\mathbf{y} - \boldsymbol{\imath}'\mathbf{y}^{-i} \; | i = 1, \ldots, n\}$, which is very similar to the problem (2) in Section 3.1. Removing i^* from the network $\mathbf{g}$ has the highest overall impact on the aggregate equilibrium contribution. The *intercentrality* of player i of parameter a in $\mathbf{g}$, where $\mathbf{G}$ can be *non-symmetric*,[23] is defined by

$$c_i(\mathbf{g}, a) = \frac{\left[\sum_{j=1}^{n} m_{ji}(\mathbf{g}, a)\right] \times h_i(\mathbf{g}, a)}{m_{ii}(\mathbf{g}, a)}. \tag{8}$$

Obviously, when $\mathbf{G}$ is symmetric as in our game, then $m_{ji}(\mathbf{g}, a) = m_{ij}(\mathbf{g}, a)$, thus the intercentrality measure reduces to $c_i(\mathbf{g}, a) = h_i(\mathbf{g}, a)^2/m_{ii}(\mathbf{g}, a)$. While KB centrality of actor i counts the number of direct and indirect paths in $\mathbf{g}$ stemming from i, the "intercentrality counts the total number of such paths that hit i; it is the sum of i's Bonacich centrality and i's contribution to every other player's Bonacich centrality" (Ballester et al., 2006, p. 1411). Theorem 3 in BCZ (2006) shows that the key player i^* has the highest intercentrality, i.e., $c_i^*(\mathbf{g}, a) \geq c_i(\mathbf{g}, a)$ for all $i = 1, \ldots, n$. In their Example 1, the authors show that the most central player (i.e., with the highest KB centrality) is not the key player for relatively large a. This follows since then indirect effects matter and, as the intercentrality takes into account both a player's centrality and his contribution to the centrality of the others, key player with the highest joint direct and indirect effect on aggregate outcome might be very well other than the most central player.

Now consider the relation of the key player problem to the key sector problem discussed is Section 3.1. We have defined the (general) factor worth and its particular

[22] In fact, Bonacich (1987) defines the network centrality measure by the vector $\mathbf{t}(\mathbf{g}, a, b) = b(\mathbf{I} - a\mathbf{G})^{-1}\mathbf{G}\boldsymbol{\imath}$, where the parameter b "affects only the length of the vector $[\mathbf{t}(\mathbf{g}, a, b)]$"(p. 1173). It is not difficult to show that $\mathbf{h}(\mathbf{g}, a) = \boldsymbol{\imath} + a\mathbf{t}(\mathbf{g}, a, 1)$. This measure is directly related to the status measure $\mathbf{k}(\mathbf{g}, a) = (\frac{1}{a}\mathbf{I} - \mathbf{G})^{-1}\mathbf{G}\boldsymbol{\imath}$, introduced by Katz (1953), since $\mathbf{k}(\mathbf{g}, a) = a\mathbf{t}(\mathbf{g}, a, 1) = \mathbf{h}(\mathbf{g}, a) - \boldsymbol{\imath}$.

[23] Non-symmetric matrices play crucial role in networks of directional relations.

case of output worth of sector i, respectively, as

$$\omega_i^{\pi}(\mathbf{A},\mathbf{f},\boldsymbol{\pi}) = \frac{m_i^{\pi}x_i}{b_{ii}}, \quad \text{and} \quad \omega_i^{o}(\mathbf{A},\mathbf{f}) = \frac{m_i^{o}x_i}{b_{ii}}.$$

The matrix $\mathbf{M}(\mathbf{g},a) = (\mathbf{I}-a\mathbf{G})^{-1}$ has exactly the same properties as the Leontief inverse, $\mathbf{B} = (\mathbf{I}-\mathbf{A})^{-1}$, where the core of the key player analysis is the adjacency matrix $\mathbf{G}$ of the network $\mathbf{g}$, while that role in the key sector identification is assigned to the input matrix $\mathbf{A}$. First, comparing the above definitions with the intercentrality measure (8), note that the diagonal elements of the matrices $\mathbf{B}$ and $\mathbf{M}(\mathbf{g},a)$ appear in their denominators. Recall that b_{ii} indicates the own gross output that is required directly and indirectly for one unit of final demand of sector i, and $m_{ii}(\mathbf{g},a)$ is the total (direct and indirect) number of paths in $\mathbf{g}$ that start at i and end at i, or, equivalently, the number of self-loops of player i. The first terms in the numerator of the factor and output worths are, respectively, the factor (say, income) and output multipliers, $m_i^{\pi} = \sum_{j=1}^{n}\pi_j b_{ji}$ and $m_i^{o} = \sum_{j=1}^{n} b_{ji}$ that are mathematically very similar to the first term in the numerator of (8), $\sum_{j=1}^{n} m_{ji}(\mathbf{g},a)$, which indicates the total number of paths from all players that end at position i. The second term in the numerator of (8) is the KB centrality $h_i(\mathbf{g},a) = \sum_{j=1}^{n} m_{ij}(\mathbf{g},a)$, while the corresponding component for both factor worth and its counterpart of output is the gross output of sector i, $x_i = \sum_{j=1}^{n} b_{ij}f_j$. The last terms will be mathematically more similar if gross output would be expressed per unit of final demands of all sectors, i.e., $f_i = 1$ for all i.[24]

We have shown that the factor worth is mathematically equivalent to the concept of the intercentrality measure. This is not surprising since the key sector problem and key player problem address conceptually the same issue, finding a sector or an actor that has the largest overall impact on the aggregate outcome.[25] Of course, the interpretations are totally different given their underlying theoretical frameworks. In the analysis of social networks directionality of relations between players and valuation of those links generates four basic types of social network data: binary nondirected, binary directed, valued nondirected, and valued directed ties. Binary data indicates only the presence or absence of ties between pairs of actors, while valued data besides presence or absence of a tie also quantify the intensity or frequency of interactions. In nondirected graph the relation is mutual, while in digraph the relation is not always reciprocal, hence the origin and the end of links are distin-

[24] If we would have parameter α_i in the utility function (7) placed before y_i, then the *weighted* KB centrality measure is defined as $\mathbf{h}_{\boldsymbol{\alpha}}(\mathbf{g},a) = (\mathbf{I}-a\mathbf{G})^{-1}\boldsymbol{\alpha}$, so that the role of the final demand vector $\mathbf{f}$ in the IO framework would be taken by the vector $\boldsymbol{\alpha}$ in the network game (BCZ, 2006, Remark 1).

[25] Temurshoev (2008) extends the study of BCZ (2006) from the key player search to a key group search.

guished. In our IO analysis of the key group, however, the network of industries is always represented by valued directed data (and possibly graphs).

4.2 The link to the coalitional game literature

A question of a *fair* allocation of gains obtained from cooperation among several actors was one of the main focus at the outset of game theory. The setup is simple: cooperation of actors results in a certain overall gain that has to be divided among the actors within the coalition. However, the last is not a trivial issue given that actors have different contributions to the coalition. The legitimate question is then how to allocate fairly the gain from cooperation to its participants. Or in other words, how important each actor is to the coalition, and what payoff they deserve? One approach is to use a *Shapley value*, named in honor of Lloyd Shapley, who introduced it in his classical 1953 paper "A value for n-person games". Using an axiomatic approach, Shapley constructed a solution remarkable for its intuitive definition and unique characterization by a set of reasonable axioms. The specialization of Shapley value to simple games[26] is often used as an index of voting power and is known as the Shapley-Shubik power index (Shapley and Shubik, 1954). Other related indicator is a Banzhaf power index proposed in Banzhaf (1965).[27] The generalization of Shapley and Banzhaf values to *coalitional structure*, where the interaction between players is *not* symmetric in a sense that actors may be part of different groups, which might make negotiations between groups impossible, is studied, in particular, by Aumann and Drèze (1974) and Owen (1977, 1981).[28] A *share function* solution of van der Laan and van den Brink (1998) assigns every player its share in the worth of the grand coalition, and contains the Shapley share function and the Banzhaf share function as special cases. Solution in terms of share functions for games with coalitional structure is introduced in van der Laan and van den Brink (2002). Since all these values are closely related to the original contribution of Shapley (1953), in what follows, we will only discuss the link of the factor worth to the Shapley value.

Formally, a *coalitional form game* on a finite set of players $N = \{1, 2, \dots, n\}$ is a function v from the set of all coalitions 2^N to the set of real numbers $\mathbb{R}$, with the properties

[26]See Shapley (1962) for details on simple games.

[27]Straffin (1977, 1988) interprets the Shapley-Shubik and Banzhaf indices as the probabilities of affecting the voting outcome. In particular, it is shown that the Shapley-Shubik index is more appropriate measure of probability of a voter influence when voters' decisions are correlated (eg., a society judging welfare by common standards), while the Banzhaf index is more appropriate if voters behave independently of each other.

[28]Cooperative games with coalitional structure imply a two-level interaction between the players (see eg., Hart and Kurz, 1983). Firstly, the value of the grand coalition is distributed amongst the coalitions, and secondly, the worth of each coalition is allocated amongst the players within this coalition. See also, Winter (1989, 1992); Owen and Winter (1992).

1. $v(\varnothing) = 0$,
2. $v(S \cup T) \geq v(S) + v(T)$, whenever $S \cap T = \varnothing$.

Interpretation of $v(S)$ is the *expected total payoff* (gain or rent) that the coalition S can get in the game v. The second property, so called superadditivity condition, implies that cooperation can only benefit players, and never makes them worse off. The Shapley value (ϕ) is one way to distribute the total gain to all players, and assigns to each game v a vector of payoffs $\boldsymbol{\phi}(v)' = (\phi_1, \phi_2, \ldots, \phi_n)$ in $\mathbb{R}^n$. Alternatively, one can think of $\phi_i(v)$ as the measure of i's power in the game v. For all $S \subseteq N$ and all $i \in S$, the marginal contribution of player i to coalition S in game v is defined by $v(S) - v(S \setminus \{i\})$. Shapley constructed the following value that assigns an expected marginal contribution of each player in the game with respect to a uniform distribution over the set of all permutations on the set of players:

$$\phi_i(v) = \sum_{S \subseteq N, i \in S} \frac{(s-1)!(n-s)!}{n!} \big[v(S) - v(S \setminus \{i\}) \big], \tag{9}$$

where s is the cardinality of S, i.e., $s = |S|$. In words, $\phi_i(v)$ is averaging *marginal contributions* of player i over the possible different permutations of coalitions, since i's marginal contribution to coalition S is, first, multiplied with the $(s-1)!$ different permutations of the members of coalition S aside from player i and the $(n-s)!$ different permutations of players outside the coalition S, then divided by the $n!$ different permutations of all the players in the grand coalition N, and the results are summed over all the coalitions S to which player i belongs.

Shapley value satisfies the following four axioms.

Efficiency: $\sum_{i \in N} \phi_i(v) = v(N)$, i.e., the resources available to the grand coalition are precisely distributed amongst all the players.

Symmetry: If $v(S \cup \{i\}) = v(S \cup \{j\})$ for every subset $S \subset N$ with $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$. That is, if players $i, j \in N$ make the same marginal contribution to any coalition S that contains neither i nor j, then i and j are symmetric with respect to game v, and have equal shares.

Dummy: If i is a dummy (or null) player, i.e., $v(S \cup \{i\}) = v(S)$ for all $S \subset N$, then $\phi_i(v) = 0$. This axiom requires that players with zero marginal contribution to every coalition are given zero payoffs.

Additivity: For any two games v and w on a set N of players, $\phi_i(v+w) = \phi_i(v) + \phi_i(w)$ for all $i \in N$, where $v + w$ is the game defined by $(v+w)(S) = v(S) + w(S)$. This axiom requires that the value is an additive operator on the space of all games.

The remarkable finding of Shapley (1953) is that there exist a *unique* value that satisfies these four simple axioms, and it is the Shapley value given in (9).[29] Now

[29] For more details see e.g., Roth (1988); Winter (2002).

we are in a position to compare the factor worth $\omega_i^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = m_i^\pi x_i / b_{ii}$ with the Shapley value. These two indicators are similar in the sense that both assess the power of an agent on the base of its marginal contribution. The difference, however, is that the factor worth focuses on the marginal contribution of a sector to the total factor generated by production sectors taken *altogether* (see (2)), while the Shapley value takes into account the marginal contributions of a player to *all permutations* on the set of players.

To see clearly the similarities and distinctions, we will check weather the factor worth satisfies the above mentioned four axioms. In the framework of the IO analysis the value of all industries is the resulting aggregate factor, $\boldsymbol{\pi}'\mathbf{x}$. It can be shown that $\sum_{i=1}^n \omega_i^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) > \boldsymbol{\pi}'\mathbf{x}$ (see Appendix), that is, the sum of the individual factor worths of all sectors is strictly larger than the total factor that all the industries generate/use. In other words, the factor worth does *not* satisfy the efficiency axiom in the context of the coalitional game. The symmetry property, however, holds in the key sector framework, which is an expectable outcome. Two sectors i and j with the same individual contributions to overall factor have identical factor worths, which follows from the fact that $\omega_i^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = m_i^\pi x_i / b_{ii} = \boldsymbol{\pi}'(\mathbf{x} - \mathbf{x}^{-i}) = \boldsymbol{\pi}'(\mathbf{x} - \mathbf{x}^{-j}) = m_j^\pi x_j / b_{jj} = \omega_j^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$. Although, factor worth satisfies the third axiom of dummy, it does not make sense to have a null sector in the IO framework. When $\boldsymbol{\pi}'(\mathbf{x} - \mathbf{x}^{-i}) = 0$, it necessarily implies that sector i does not purchase inputs from any sector in the economy and does not supply intermediate inputs to all domestic industries. Therefore, the null sector would be a sector that buys all its intermediate inputs from abroad, and provides its output to the final demand categories only, which is entirely unrealistic case. Finally, Section 3.3 on the combined key sector problem implies that the additivity property of the Shapley value is also satisfied in case of the factor worth. If we would define the combined factors worth by $\omega_i^{v,l}(\mathbf{A}, \mathbf{f}, \mathbf{v}, \mathbf{l})$, where $\mathbf{v}$ and $\mathbf{l}$ are the direct sectoral value-added and labor coefficients, respectively, then from the factor worth definition it follows that $\omega_i^{v,l}(\mathbf{A}, \mathbf{f}, \mathbf{v}, \mathbf{l}) = \omega_i^v(\mathbf{A}, \mathbf{f}, \mathbf{v}) + \omega_i^l(\mathbf{A}, \mathbf{f}, \mathbf{l})$, which is the additivity condition in the context of the coalitional game literature. All in all, we have shown that the Shapley value and the factor worth are intuitively closely related, but their applications are quite different given that they are the outcome of the two entirely different frameworks.

5 Application to the Australian economy

We have already noted that the input-output linkage studies (implicitly) accepted the k sectors (where $1 < k < n$) with the largest individual factor worths as the *key group* of k sectors. In this section by an example of the Australian economy we show that this is not true as long as the HEM approach is concerned, i.e., the k sectors with the highest factor worths, in general, do *not* compose the *key group* of size k.

We have used data from Foran et al. (2005) and Centre for Integrated Sustainability Analysis (2005) that include the 1994-1995 Australian IO tables and satellite accounts at 136 industry-level classification.[30] For simplicity, the industries were codified, whose list is given in Table 4 in the Appendix. The key sector/group problem is performed for two environmental, one financial and one social factors, which are, respectively, water use, carbon dioxide (CO_2) emissions, gross operating surplus, and wages and salaries. The results are reported in the first five columns of Table 1 in terms of *relative* group factor worths, i.e., the group factor worths as a percentage of the overall factor use/generation before the extraction of sectors comprising the group. For instance, the relative profits (gross operating surplus) worth of sectors i and j ($\neq i$) equals $(\omega^p_{i,j}(\mathbf{A}, \mathbf{f}, \mathbf{p})/\mathbf{p}'\mathbf{x}) \times 100$, where $\mathbf{p}$ is the vector of sectoral direct profits coefficients, thus $\mathbf{p}'\mathbf{x}$ is the total gross operating surplus in the economy. Hence, these relative measures refer to the percentage decrease in economy-wide factor use/generation caused by the extraction. We only report the top 5 groups of size $k \in [1, 4]$, where obviously the group with rank 1 in each list is the corresponding *key group*.

Several observations can be made from Table 1. The first and most obvious observation is that different objectives give different composition of the key group of certain size and different rankings of sectors or group of sectors. This is totally expectable, as different sectors perform different functions in the economy, thus should not be equivalent it terms of their contributions to the overall consumption/production of various factors.

Second outcome is that the composition of the key group of size k is, in general, different from the k sectors with the largest (individual) factor worths, which confirms our expectation that the key sector problem is not equivalent to the key group problem. For example, let us look at the key group problem in terms of water use. The second column of Table 1 shows that Dairy cattle & milk (Dc) is the key sector in water use with the relative water consumption worth of 19.5%.[31] The key group of size two consists of the key sector Dc and Beef cattle (Bc) jointly accounting for 37.6% of the economy-wide water consumption, which, however, does *not* include Diary products (Dp) that has the second largest water (usage) worth. Further, the key group of size 3 besides Dc and Bc includes Water supply, sewerage and drainage services (Wa), which has only the sixth rank according to the key sector problem

[30] Foran et al. (2005) give detail description of the data sources and its construction.

[31] In the language of the HEM problem, if Dairy cattle & milk (Dc) sector would be eliminated from the economy then the overall use of water would be reduced by 19.5%.

Table 1: Relative group factor worths of Australian industries, 1994-1995

Rank	Group of size k and its relative factor worth (%)				Factor multipliers	Factor use/ generation	Factor responsibility
	$k=1$	$k=2$	$k=3$	$k=4$			
	Objective: Water use				(ths.l/A$)	(Tl)	(Tl)
1 (key)	Dc (19.5)	Bc, Dc (37.6)	Bc, Dc, Wa (48.1)	Bc, Dc, Vf, Wa (58.0)	Ri (7.47)	Dc (3.54)	Dp (2.89)
2	Dp (18.6)	Dc, Mp (37.3)	Dc, Mp, Wa (47.7)	Dc, Mp, Vf, Wa (57.5)	Sc (1.64)	Bc (3.23)	Mp (2.68)
3	Bc (18.2)	Bc, Dp (36.8)	Bc, Dc, Vf (47.6)	Bc, Dp, Vf, Wa (57.1)	Dc (1.48)	Wa (2.02)	Fd (1.35)
4	Mp (18.1)	Dp, Mp (36.4)	Bc, Dp, Wa (47.3)	Dp, Mp, Vf, Wa (56.7)	Su (1.26)	Vf (1.80)	Ho (1.13)
5	Vf (10.7)	Dc, Wa (30.0)	Dc, Mp, Vf (47.2)	Bc, Dc, Fd, Wa (55.9)	Bc (0.73)	Ri (1.43)	Wa (1.12)
	Objective: CO_2 emissions				(kg/A$)	(Mtonnes)	(Mtonnes)
1 (key)	El (32.8)	Bc, El (52.9)	Bc, Fr, El (64.4)	Bc, Fr, El, Is (69.0)	Fr (98.3)	El (136.6)	Mp (59.7)
2	Bc (20.3)	El, Mp (50.6)	Fr, Mp, El (62.1)	Bc, Fr, El, Wt (67.7)	Sw (25.2)	Bc (81.2)	El (53.8)
3	Mp (18.3)	El, Fr (44.9)	Bc, Is, El (57.6)	Bc, Fr, El, Rb (67.5)	Bc (17.9)	Fr (50.9)	Fr (38.0)
4	Fr (12.3)	El, Is (37.5)	Bc, El, Wt (56.4)	Bc, Fr, El, At (67.2)	Hw (15.4)	Is (17.9)	Rt (21.7)
5	Is (5.5)	El, Wt (36.4)	Bc, El, Rb (56.3)	Bc, Fd, Fr, El (67.1)	Lm (14.8)	At (10.1)	Rb (16.8)
	Objective: Gross operating surplus (profits)				(A$/A$)	(A$ Bln)	(A$ Bln)
1 (key)	Dw (21.7)	Dw, Wt (31.2)	Dw, Rb, Wt (37.1)	Dw, Rb, Rt, Wt (42.5)	Dw (0.84)	Dw (38.7)	Dw (41.6)
2	Wt (9.7)	Dw, Rb (27.9)	Dw, Rt, Wt (36.7)	Dw, Nb, Rb, Wt (41.7)	Si (0.68)	Wt (7.5)	Rb (11.9)
3	Rb (6.6)	Dw, Rt (27.5)	Dw, Nb, Wt (35.9)	Dw, Nb, Rt, Wt (41.3)	Bl (0.63)	Rb (7.1)	Rt (11.0)
4	Rt (5.9)	Dw, Ms (26.9)	Dw, Ms, Wt (35.1)	Dw, Ho, Rb, Wt (40.8)	Br (0.622)	St (6.44)	Wt (9.4)
5	Ms (5.3)	Dw, Nb (26.7)	Dw, Ho, Wt (35.0)	Dw, Ms, Rb, Wt (40.8)	Ng (0.62)	Ms (6.39)	Nb (9.0)
	Objective: Net wages and salaries				(A$/A$)	(A$ Bln)	(A$ Bln)
1 (key)	Wt (12.4)	Rt, Wt (22.8)	Hs, Rt, Wt (31.7)	Ed, Hs, Rt, Wt (40.5)	Ed (0.61)	Ed (14.6)	Rt (18.3)
2	Rt (10.9)	Hs, Wt (21.3)	Ed, Rt, Wt (31.6)	Gv, Hs, Rt, Wt (39.0)	Gd (0.58)	Hs (14.2)	Hs (15.5)
3	Hs (9.1)	Ed, Wt (21.24)	Ed, Hs, Wt (30.2)	Gv, Ed, Rt, Wt (38.8)	Hs (0.533)	Rt (11.7)	Ed (14.5)
4	Ed (9.09)	Hs, Rt (20.1)	Gv, Rt, Wt (30.1)	Hs, Nb, Rt, Wt (38.0)	Os (0.53)	Wt (11.6)	Gv (11.5)
5	Gv (7.8)	Ed, Rt (20.0)	Ed, Hs, Rt (29.1)	Ed, Nb, Rt, Wt (37.9)	Gv (0.50)	Gv (10.0)	Nb (10.8)
Total	136	9,180	410,040	13,633,830	136	136	136

Note: "Total" is the total number of all possible groups of size k. Mathematically, it is equal to the combinations of $n = 136$ sectors taken k at a time, $C_k^n = n!/(k!(n-k)!)$. One teraliter (Tl) is equivalent to 10^{12} litres. The source of the seventh column "Factor use/generation" is the satellite accounts in Foran et al. (2005) and Centre for Integrated Sustainability Analysis (2005), while the rest are own computations based on these data. One megatonnes (Mtonnes) equals 10^6 tonnes. Sectors' abbreviations are listed in Table 4.

with water worth of 10.6% (not shown in Table 1). The traditional "top-list" approach would consider the "key" group of size 4 consisting of dairy and beef cattle, and dairy and meet products (i.e., Dc, Dp, Bc and Mp as the top 4 sectors with the largest individual water usage worths), while the formal key group problem finds beef and diary cattle (Bc, Dc), Vegetable and fruit growing (Vf), and Water supply, sewerage & drainage (Wa) to be the part of the key group. The legitimate question is why the "top-list" approach does not give the true outcome identified by the key group problem.[32] The group factor worth of sectors $i_1, \ldots, i_k$ can be rewritten as

$$\omega^{\pi}_{i_1,\ldots,i_k}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \sum_{s=i_1}^{i_k} \pi_s x_s + \sum_{j \neq i_1,\ldots,i_k} \pi_j \left(x_j - x_j^{-\{i_1,\ldots,i_k\}} \right),$$

which shows that the factor worth of the extracted sectors includes not only their *direct* contributions to factor usage/generation (the first sum), but also their *indirect* contributions to factor consumed/produced by every other sector outside the group (the second sum).[33] Hence, with inherently different structures and sizes of inter-sectoral links, and intermediate and final demands, the group of k sectors will play quite a different role in overall factor usage/generation process than a single industry, in particular, through its indirect channel.

This result wedges a bridge between the IO linkage analysis and the sociology literature on actors' importance in social networks. This link has to do with what sociologists call a *redundancy principle* (see e.g., Burt, 1992; Borgatti, 2006), which in our framework means that sectors may be redundant with respect to their linkage patterns, factor generation abilities and final demand structures. For example, sectors can be redundant when they connect the same third industries to each other, or when they are connected to the same third parties, in both cases with approximately the same sizes of inter-industry transactions, factor generation, and final demands. Sectors are called to be *structurally equivalent* in the latter case of redundancy in the sociological terminology. In the framework of social networks, Temurshoev (2008) extended a notion of *intercentrality measure* introduced by Ballester et al. (2006) in identifying a *key player* from social planner perspective to a *group intercentrality measure*, and showed that there is a link between the key group members (that

[32]Note that in our example these two approaches give identical results for $k \in [1, 4]$ when the objectives are profits, and wages and salaries. We should, however, stress that these observations by no means can subside the existence of the difference between the two approaches, and thus the key group problem should always be given preference over the "top-list" approach whenever the HEM is the study methodology. Application to the Kyrgyzstan economy for value-added and gross output also confirm the difference between the "top-list" and the key group problem approaches in defining the key group. These results are shown in the Appendix.

[33]Note that in the case of gross output being the objective, i.e., when $\pi_i = 1$ for all i, the group output worth equals the sum of gross outputs of the extracted sectors and their indirect contributions to every other sectors' gross outputs.

are ex ante identical) and clusters of similar agents, where clusters are identified by a hierarchical agglomerative cluster analysis. That is, the key group generally contains members from different clusters, i.e., key group members are rather nonredundant with respect to the patterns of ties to their alters provided that the agents are ex ante identical. We believe that namely this redundancy principle in the IO framework explains the fact that Dairy products (Dp) that ranks high in the key sector problem (i.e., for $k = 1$) is not contained in key groups of size $k > 1$ in Table 1 in the case of water usage. For example, key group of size 2 contains dairy cattle and beef cattle (Dc and Bc) and not the second largest consumer of water - Dairy products (Dp), simply because dairy cattle and dairy products (Dc and Dp) have rather similar patterns of production linkages, water usage and final demand structure than those of dairy cattle and beef cattle (Dc and Bc).[34]

The third observation from Table 1 is that sectors in the key group of size k are also part of the key group of size $k + 1$, which raises a question of whether this is a general property or is a mere coincidence. It turns out that this is *not* true in general, i.e., the group target selection problem is not equivalent to a *sequential* key sector problem.[35] (The author can supply an interested reader by a hypothetical IO table that confirms the last statement.) One might (rightly) think that this fact is unfortunate from computational perspective, since this urges an analyst to compute the factor worths for *all* possible combinations of k from all n sectors, which, for instance, in our case with group of size 4 required to consider more than 13.6 million combinations,[36] and that search process would be significantly reduced (i.e., to only 133 cases) if the key group problem and the sequential key sector problem would be equivalent. Given that we have conjectured that the key group members are rather

[34]This can be proved formally using cluster analysis, which is, however, beyond the scope of this paper. In this respect, our study has a link to Hoen (2002), who analyzes the groups of sectors with strong connections using different cluster identification methods and ends at choosing a *block diagonalization method* to suit best for clustering purpose. This method rearranges sectors in such a way that the important linkages (bigger than some specified threshold level) of a matrix (such as the intermediate values, input coefficients, Leontief inverse) appear in blocks along the main diagonal, and thus sectors in one block comprise one separate cluster. However, a word of caution is in place with respect to diagonalization method: it does *not* allow for "cluster switching". For instance, Howe and Stabler (1989) showed that an object may be assigned to totally different cluster if the number of identified clusters changes. In fact, this property of block diagonalization Hoen (2002) considers positively as other "cluster methods ... did not show this phenomenon [i.e., cluster switching] for sectors" (p. 139). However, the HEM allows for sector switching if one interprets the key group members in terms of different clusters' membership, at least, theoretically (see the next observation).

[35]By a sequential search we mean the following: once the key group of size k has been identified, one needs only to add to this group an extra sector from all possible $n - k$ remaining industries in order to identify the key group of size $k + 1$.

[36]This computation in a PC with a memory (RAM) of 4 GB and a Windows Experience Index base score of 4.6 took overall 17 minutes and 44 seconds. The MATLAB program can be provided by the author upon request. However, we should note that the time mentioned *might* be reduced if one is capable of writing a more efficient program for such computation.

nonredundant with respect to their linkage patterns, factor generation capability and final demand structure, and, thus, should be part of different clusters with similar impacts, this result allows, at least theoretically, for "cluster switching" of sectors between clusters once the number of (identified) clusters changes.[37] The phenomenon of "cluster switching" have been found, for example, in Howe and Stabler (1989). Hence, the fact that the key group problem requires to search for all possible combinations is, in fact, advantageous as it reveals cases of "cluster switching" if they do exist.

The forth observation is that in Australia in 1994-1995 a group of few industries accounted for the majority of the environmental factors, while generation of profits and salaries is relatively dispersed among sectors. So 58% and 53% of, respectively, water (direct and indirect) consumption and CO_2 emissions are due to the key groups of size 4 and 2 from the total of 136 sectors. This has, for example, the following policy implication: focusing on a very few industries would give quite a big impact in terms of, say, CO_2 emissions, but in order to have large effect on social factors generation many more industries should be given some policy priority. More specifically, we can see that Electricity supply (El) alone accounts for 32.8% of the Australian carbon dioxide emissions, while other factor worths (i.e., for water use, profits and wages generation) of key sectors are much smaller. Only Beef cattle (Bc) and Electricity supply (El) (members of the key group of size 2) generate 52.9% of Australian CO_2 emissions, hence any attempt to reduce carbon emissions should target these industries in the first place (say, by encouraging reduction of CO_2 emission intensity in these sectors, or as suggested by Daniels (1992, see fn. 18), Australia has to re-direct its production from these high emissions-intensive industries towards more value-adding sectors). Our analysis is complementary to Lenzen's (2003) study, who in his IO structural path analysis finds electricity generation for private final consumption as the most important path (of zero order) in producing CO_2 emissions (see Table 3, p. 22), and second in his list is the second-order path originating from Beef cattle (Bc) to Meat products (Mp) for exports, since "Beef cattle for exported meat product ... alone ... is responsible for ... 37.5 Mt CO_2-e of greenhouse gases..." (p. 27). Note that Meat product (Mp) is a member of, at least, groups of sizes 2 and 3 with the second largest group CO_2 emissions worths in Table 1. In case of water use, the key sector is Dairy cattle (Dc), while Beef cattle (Bc), Dairy cattle (Dc)

[37] To see this consider the following hypothetical case with four sectors. Suppose there are three clusters: {1,2}, {3} and {4} and the key group of size three is {1,3,4}. It might very well happen that in reducing the number of clusters we get the following two clusters: {1,3} and {2,4}, in which case sector 2 "switches" from its original cluster {1,2} to the cluster {4}. Then, in this "cluster switching" case the key group of size two can be, for example, {1,4}. Note that in this case the key group of size 2 is a part of the key group of size 3, which does not have to be necessarily true in general.

and Water supply, sewerage and drainage services (Wa) jointly account for 48.1% of water consumption. Hence, again any policy towards more efficient use of water must consider these mentioned industries in the first place. Here, also our results are complementary to those found in Lenzen (2003) with different IO apparatus of structural path analysis, who then similarly mentions that "... in order to reduce the irrigation-induced stress on the Murray-Darling river system in South-Eastern Australia, shifts in production from water-intensive industries towards more value-adding sectors have been recommended" (p. 29). Analogous conclusions can be made with respect to the two other factors of profits and wages.

The last observation is that the percentage decrease in overall factor usage/ production upon extraction of groups is always smaller than the sum of the individual relative factor worths of sectors comprising the group. This mathematically is equivalent to $\sum_{s=1}^{k} \omega_{i_s}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) > \omega_{i_1,i_2,\ldots,i_k}^{\pi}(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi})$ for all $k = 2, 3, \ldots, n$ (the proof is available from the author upon request), which exactly reflects the redundancy principle in the IO framework discussed above. Note that, however, summation of percentages of the relative factor worths for given k does not have an economic meaning in itself as these relative worths do *not* sum up to 100% due to the fact that each sector's (or group's) contribution is examined under the assumption that the rest of the sectors are active.

In order to compare the results of the generalized HEM to other indicators, in the last three columns of Table 1 we present the top 5 sectors with the largest factor multipliers, direct factor usage/generation, and factor responsibility. The first two indicators do not need additional explanation, hence we briefly discuss the third one. Multiplying the diagonalized matrix of the factor coefficients vector by the Leontief inverse gives the matrix $\hat{\boldsymbol{\pi}}\mathbf{B}$, whose ij-th element shows the amount of factor used/produced by sector i per unit final demand of sector j. Hence, the ij-th entry of the matrix $\hat{\boldsymbol{\pi}}\mathbf{B}\hat{\mathbf{f}}$ is the amount of the factor used/generated by sector i due to final demand of sector j, or equivalently, how much factor was consumed/produced by sector i *for* sector j. Thus, summing over all i's gives the amount of the factor consumed/produced by all industries *for* sector j, which is the j-th element of the vector $\boldsymbol{\pi}'\mathbf{B}\hat{\mathbf{f}}$. In other words, this is the amount of the factor that sector j is responsible for, hence the term "responsibility".[38]

Multipliers are traditionally used to identify the importance of sectors. Table1, however, shows that factor multipliers can give quite different results than those based on the HEM approach. This is expectable since factor worths besides the size of multipliers also take into account sectors' gross outputs size and net input dependencies. Rice (Ri) has the highest water use multiplier (7470 litres per A$ of

[38]See Hoen and Mulder (2003) for similar computation in analyzing the Dutch CO_2 emissions.

its final demand), while it is not a member of the key groups of size $k \in [1, 4]$, and, moreover, it does not show up in the list of the top 5 groups at all. Rice (Ri) though is the 5-th largest direct consumer of water (1.43 Tl), it is not in the list of the top 5 responsible sectors. In case of CO_2 emissions, Forestry (Fr) has the largest CO_2 multiplier, but it is not a member of the key group of size $k < 3$. For gross operating surplus all four indicators give quite close outcomes with Ownership of dwellings (Dw) being the most important sector in all respect. Education (Ed) has the largest wages multiplier, and becomes a member of the key group of size 4. All in all, these results do not mean that factor multipliers are useless from policy perspective. The advantage of multipliers lies in the price evaluation of commodities as multipliers are expressed per unit of final demand. In other words, industries with high factor multipliers are sensitive to changes in the factor price (see e.g., Dietzenbacher and Velázquez, 2007). In our case a pricing policy that tries to internalize the costs of using water and CO_2 emissions will have the largest impact on the prices of, respectively, Rice (Ri) and Forestry (Fr).[39]

Notice also that for water use and CO_2 emissions there is a perfect correspondence between the key group members and the list of sectors with the largest direct factor usage/generation in Table 1 . But this is not always the case: the largest capacity of generating wages has Education (Ed, 14.6 Bln A$), which is not a member of the key group of size $k < 4$. Instead, Retail trade (Rt), which is *responsible* for the largest amount of wages (18.3 A$), is part of the key group of size $k \geq 2$. For water usage and CO_2 emissions Dairy products and Meat products (Dp and Mp) are respectively the most responsible sectors, while in both cases they do not show up as a part of the key groups. However, these industries are members of groups that are second in the list. All in all, it seems that the HEM approach takes into account both sectors' direct factor consumption/generation and sectors' responsibility in using/generating the factor by other industries. This is, of course, the specific advantage of using the generalized HEM proposed in this paper, which fully considers all kinds of interlinkages associated with the hypothetically extracted sector(s).

6 Conclusion

In this paper we investigated the issue of identification of a key sector and a key group of sectors in the economy by a complete hypothetical extraction method

[39] In this respect Foran et al. (2005) regarding agricultural, forestry and food products in Australia state: "... the prices we pay for the products reflect the marginal cost of production, rather than the full resource and environmental costs of production. ... Moves to internalize the full costs of production in the final price of the market product may mean substantial price increases" (p. 1).

(HEM). We show that for this purpose the analyst does not have to perform the three step procedure of the HEM: delete the corresponding row(s) and column(s) of the input matrix, calculate the overall factor usage/production in the hypothetical case, and find the difference between the actual and hypothetical objectives. These steps are rather excessive given that we have found quite simple formulas (measures of industries' factor worths) in getting the desired outcome.

We showed that the key sector problem and the key group problem have, in general, different solutions. This is demonstrated in the empirical applications of the mentioned problems to the Australian economy for four factors of water use, CO_2 emissions, profits, and wages and salaries, and to the Kyrgyzstan economy for total income and gross output that is given in the Appendix. In general, the key group has the highest group factor worth, which is directly related to the overall impact on aggregate factor usage/generation of an incremental changes in direct self-dependencies of the sectors comprising the group, and inversely related to own net input dependence of the group members. The last interpretation is the result of linking the HEM to the fields of influence method. It is proved that an increase (resp. decrease) in an input coefficient never decreases (resp. increases) the factor worth of any sector. In both cases the necessary and sufficient condition for a strict change is that the sector supplying more per unit depends directly and/or indirectly on a sector whose worth is going to change.

In case of Australia, we found that in 1994-1995 out of 136 industries Beef cattle and Electricity supply jointly generated 52.9% of total (direct and indirect) CO_2 emissions, while Beef cattle, Dairy cattle, and Water supply, sewerage and drainage services jointly accounted for 48.1% of overall water consumption. Hence any attempt to reduce carbon emissions and more efficient use of water in Australia should target these industries in the first place.

Appendix

A. Key group vs. key sector in the Kyrgyzstan economy

We use 1997 IO table in current prices of the Kyrgyzstan economy (constructed by the National Statistical Committee of Kyrgyz Republic) that has 34 industry classifications (see Table 3). The results of the key sector/group problems in Table 2 are again given in terms of *relative* group income and output worths, thus measure the percentage decrease in economy-wide income and output caused by the extraction.

Table 2: Relative group income and output worths of sectors in Kyrgyzstan, 1997

Ranking	Group of size k and its relative income and output worth (%)				
	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
	Objective: national income				
1 (key)	1 (43.99)	{1,22} (55.47)	{1,20,22} (62.18)	{1,4,20,22} (67.93)	{1,4,16,20,22} (72.33)
2	22 (13.34)	{1,20} (51.02)	{1,4,22} (61.25)	{1,16,20,22} (66.70)	{1,4,20,22,30} (72.20)
3	6 (11.89)	{1,4} (49.93)	{1,16,22} (60.22)	{1,12,20,22} (66.54)	{1,4,6,20,22} (71.94)
4	20 (7.70)	{1,28} (49.55)	{1,12,22} (59.88)	{1,20,22,30} (66.44)	{1,4,20,22,28} (71.82)
5	28 (6.21)	{1,16} (48.0)	{1,22,30} (59.74)	{1,6,20,22} (66.20)	{1,4,7,20,22} (71.48)
	Objective: gross output				
1 (key)	1 (32.88)	{1,22} (43.03)	{1,12,22} (52.01)	{1,12,20,22} (60.44)	{1,6,12,20,22} (67.53)
2	6 (13.48)	{1,12} (42.05)	{1,20,22} (51.56)	{1,6,12,22} (59.14)	{1,12,20,22,29} (65.72)
3	22 (11.67)	{1,20} (41.71)	{1,12,20} (50.78)	{1,6,20,22} (58.69)	{1,7,12,20,22} (64.97)
4	20 (9.34)	{1,6} (40.83)	{1,6,22} (50.20)	{1,6,12,20} (58.63)	{1,4,6,20,22} (64.54)
5	12 (9.27)	{1,4} (38.92)	{1,6,12} (49.96)	{1,4,20,22} (57.43)	{1,12,16,20,22} (64.31)
Total	34	561	5984	46376	278256

"Total" is the total number of all possible groups of size k. Mathematically, it is equal to the combinations of $n = 34$ sectors taken k at a time, $C_k^n = n!/(k!(n-k)!)$.

Table 3: Sectoral classification of the Kyrgyzstan 1997 IO Table

No.	Industry description	No.	Industry description
1	Agriculture and hunting	18	Steam and hot water supply
2	Forestry and fishing	19	Water generation, purification and distribution
3	Coal, crude oil and gas production	20	Construction
4	Ore extraction	21	Wholesale trade
5	Other minerals (mining) industries	22	Retail trade
6	Food, beverages, and tobacco	23	Car sale and servicing of private & house use goods
7	Textile and leather manufacture	24	Hotels and restaurants
8	Wood and wood products	25	Transportation, subsidiary transport activities
9	Paper, printing and paper products	26	Post and communication services
10	Chemicals, rubber & plastic manufacturing	27	Finance
11	Other non-metallic mineral products	28	Operations with real estate, rent & business services
12	Metallurgy industry	29	Government administration
13	Finished metallic products	30	Education
14	Machinery and equipment	31	Public health and social services
15	Other manufacturing, secondary processing	32	Environmental purity protection services
16	Electricity	33	Unions, rest, culture and sports activities
17	Gas fuel production and distribution	34	Rendering individual services

An interested reader can see that all the four observations made from the similar results on Australian economy hold also for the Kyrgyzstan case. Notice, however, that the structure of these countries are entirely different. For instance, the key sector in generating overall income in Kyrgyzstan in Agriculture and hunting (sector 1), while the Australian key sectors in generating profits and salaries are, respectively, Ownership of dwellings (DW) and Wholesale trade (Wt). This is, of course, entirely expectable given the level of economic development in these two countries, where one is mainly agricultural based (developing) economy and the other is service-based (developed) country.

B. Proofs

Proof of Lemma 1. We give our proof within the IO framework. First note that the matrix $\mathbf{I} - \mathbf{e}_i\mathbf{e}_i'$ has ones in all diagonal entries except for its ii-th position, and zero otherwise. Hence, $\mathbf{A}^{-i} = (\mathbf{I} - \mathbf{e}_i\mathbf{e}_i')\mathbf{A}(\mathbf{I} - \mathbf{e}_i\mathbf{e}_i')$. We make use of the well-known formula of the inverse of a sum of matrices (see e.g., Henderson and Searle, 1981):

$$(\mathbf{X} - \mathbf{U}\mathbf{D}^{-1}\mathbf{Z})^{-1} = \mathbf{X}^{-1} + \mathbf{X}^{-1}\mathbf{U}(\mathbf{D} - \mathbf{Z}\mathbf{X}^{-1}\mathbf{U})^{-1}\mathbf{Z}\mathbf{X}^{-1}, \tag{A1}$$

$$(\mathbf{X} + \mathbf{u}\mathbf{z}')^{-1} = \mathbf{X}^{-1} - \frac{1}{1 + \mathbf{z}'\mathbf{X}^{-1}\mathbf{u}}\mathbf{X}^{-1}\mathbf{u}\mathbf{z}'\mathbf{X}^{-1}. \tag{A2}$$

Since $\mathbf{e}_i\mathbf{e}_i'\mathbf{e}_i\mathbf{e}_i' = \mathbf{e}_i\mathbf{e}_i'$ (as $\mathbf{e}_i'\mathbf{e}_i = 1$), one can easily confirm that $(\mathbf{I} - \mathbf{e}_i\mathbf{e}_i')(\mathbf{I} - \mathbf{e}_i\mathbf{e}_i') = \mathbf{I} - \mathbf{e}_i\mathbf{e}_i'$. Then using (A1) it follows that

$$\mathbf{B}^{-i} = (\mathbf{I} - (\mathbf{I} - \mathbf{e}_i\mathbf{e}_i')\mathbf{A}(\mathbf{I} - \mathbf{e}_i\mathbf{e}_i'))^{-1} = \mathbf{I} + (\mathbf{I} - \mathbf{e}_i\mathbf{e}_i')[\mathbf{A}^{-1} - (\mathbf{I} - \mathbf{e}_i\mathbf{e}_i')]^{-1}(\mathbf{I} - \mathbf{e}_i\mathbf{e}_i'). \tag{A3}$$

Using (A1) again we can write the Leontief inverse as $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + (\mathbf{A}^{-1} - \mathbf{I})^{-1}$, which implies that $(\mathbf{A}^{-1} - \mathbf{I})^{-1} = \mathbf{B} - \mathbf{I}$. This together with (A2) allows us to write the inverse in the right-hand side (rhs) of (A3) as:

$$((\mathbf{A}^{-1} - \mathbf{I}) + \mathbf{e}_i\mathbf{e}_i')^{-1} = \mathbf{B} - \mathbf{I} - \frac{1}{b_{ii}}(\mathbf{B} - \mathbf{I})\mathbf{e}_i\mathbf{e}_i'(\mathbf{B} - \mathbf{I}), \tag{A4}$$

where the last follows since $\mathbf{e}_i'(\mathbf{B} - \mathbf{I})\mathbf{e}_i = b_{ii} - 1$. Plugging (A4) in (A3) and using the fact that $\mathbf{e}_i'\mathbf{B}\mathbf{e}_i = b_{ii}$, some simple matrix multiplication yields $\mathbf{B}^{-i} = \mathbf{e}_i\mathbf{e}_i' + \mathbf{B} - \frac{1}{b_{ii}}\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\mathbf{B}$, which completes the proof. □

Derivation of problem (4)***.*** The objective function in problem (2) is $\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-i} = \boldsymbol{\pi}'(\mathbf{B}\mathbf{f} - \mathbf{B}^{-i}\mathbf{f}^{-i})$. Adding and subtracting $\mathbf{B}^{-i}\mathbf{f}$ to the expression in the brackets gives $\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-i} = \boldsymbol{\pi}'(\mathbf{B} - \mathbf{B}^{-i})\mathbf{f} + \boldsymbol{\pi}'\mathbf{B}^{-i}(\mathbf{f} - \mathbf{f}^{-i})$. It is apparent that $\mathbf{f} - \mathbf{f}^{-i} = f_i\mathbf{e}_i$. This together with Lemma 1 yields

$$\begin{aligned}
\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-i} &= \boldsymbol{\pi}'\left(\frac{1}{b_{ii}}\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\mathbf{B} - \mathbf{e}_i\mathbf{e}_i'\right)\mathbf{f} + f_i\boldsymbol{\pi}'\left(\mathbf{B} - \frac{1}{b_{ii}}\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\mathbf{B} + \mathbf{e}_i\mathbf{e}_i'\right)\mathbf{e}_i \\
&= \frac{1}{b_{ii}}\boldsymbol{\pi}'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{f} - f_i\pi_i + f_i\boldsymbol{\pi}'\mathbf{B}\mathbf{e}_i - \frac{f_i}{b_{ii}}\boldsymbol{\pi}'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{e}_i + f_i\pi_i \\
&= \frac{1}{b_{ii}}\boldsymbol{\pi}'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{f} + f_i\boldsymbol{\pi}'\mathbf{B}\mathbf{e}_i - \frac{f_i}{b_{ii}}\boldsymbol{\pi}'\mathbf{B}\mathbf{e}_i\mathbf{e}_i'\mathbf{B}\mathbf{e}_i = \frac{1}{b_{ii}}\mathbf{m}_\pi'\mathbf{e}_i\mathbf{e}_i'\mathbf{x},
\end{aligned}$$

where the last term follows since $\mathbf{e}_i'\mathbf{B}\mathbf{e}_i = b_{ii}$. □

Proof of Theorem 2. Using the definitions of the factor worth, factor multiplier and equation (1), we have $\omega_i^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \frac{1}{b_{ii}}m_i^\pi x_i = \left(\sum_{j=1}^n \pi_j b_{ji}\right)\sum_{j=1}^n \frac{b_{ij}}{b_{ii}}f_j$. Then,

$$\Delta_i^\pi \equiv \omega_i^\pi(\widetilde{\mathbf{A}}, \mathbf{f}, \boldsymbol{\pi}) - \omega_i^\pi(\mathbf{A}, \mathbf{f}, \boldsymbol{\pi}) = \left(\sum_{j=1}^n \pi_j\tilde{b}_{ji}\right)\sum_{j=1}^n \frac{\tilde{b}_{ij}}{\tilde{b}_{ii}}f_j - \left(\sum_{j=1}^n \pi_j b_{ji}\right)\sum_{j=1}^n \frac{b_{ij}}{b_{ii}}f_j,$$

where $\tilde{b}_{ij}$ is a generic element of $\widetilde{\mathbf{B}} = (\mathbf{I} - \widetilde{\mathbf{A}})^{-1}$. Adding and subtracting $\left(\sum_j \pi_j \tilde{b}_{ji}\right) \sum_j \frac{b_{ij}}{b_{ii}} f_j$ to the last expression and noting that $\tilde{m}_i^\pi = \sum_{j=1}^n \pi_j \tilde{b}_{ji}$ yields

$$\Delta_i^\pi = \tilde{m}_i^\pi \sum_{j=1}^n \left(\frac{\tilde{b}_{ij}}{\tilde{b}_{ii}} - \frac{b_{ij}}{b_{ii}}\right) f_j + \left(\sum_{j=1}^n \pi_j (\tilde{b}_{ji} - b_{ji})\right) \frac{x_i}{b_{ii}}. \tag{A5}$$

From (3) (see also fn. 11) it follows that $\tilde{b}_{ij} = b_{ij} + \epsilon_i b_{cj}$, where $\epsilon_i = \alpha b_{ir}/(1 - \alpha b_{cr})$. Therefore,

$$\frac{\tilde{b}_{ij}}{\tilde{b}_{ii}} - \frac{b_{ij}}{b_{ii}} = \frac{b_{ij} + \epsilon_i b_{cj}}{b_{ii} + \epsilon_i b_{ci}} - \frac{b_{ij}}{b_{ii}} = \frac{\epsilon_i (b_{ii} b_{cj} - b_{ci} b_{ij})}{b_{ii}(b_{ii} + \epsilon_i b_{ci})}.$$

Plugging the last expression in (A5) and using $\tilde{b}_{ij} = b_{ij} + \epsilon_i b_{cj}$ after some simple algebraic transformations we obtain

$$\Delta_i^\pi = \frac{\alpha}{b_{ii}(1 - \alpha b_{cr})} \left[b_{ir} \frac{\tilde{m}_i^\pi}{\tilde{b}_{ii}} \sum_{j=1}^n \left(b_{ii} b_{cj} - b_{ci} b_{ij}\right) f_j + b_{ci} \tilde{m}_r^\pi x_i \right]. \tag{A6}$$

The well-know property of the Leontief inverse is that $b_{ii} \geq 1$ and $b_{ii} > b_{ij} \geq 0$ for all i and all $j \neq i$. Theorem 1 in Zeng (2001) shows that $b_{ii} b_{cj} \geq b_{ci} b_{ij}$, with strict inequality holding when $j = c \neq i$. Hence, $\sum_j (b_{ii} b_{cj} - b_{ci} b_{ij}) f_j > 0$ for all $i \neq c$ (assuming that at least $f_c > 0$). It is not difficult to see that for $i = c$ every term in this sum is zero, hence the first term of Δ_c^π in (A6) (when $i = c$) vanishes, however, its second term is positive as $b_{cc} \geq 1$. So it always holds that $\Delta_c^\pi \gtreqless 0$ if $\alpha \gtreqless 0$, and this is also always the case when $i = r$, i.e., $\Delta_r^\pi \gtreqless 0$ for $\alpha \gtreqless 0$ as $b_{rr} \geq 1$. However, for all other $i \neq r, c$ the expression within the square brackets in (A6) is not always positive, and becomes zero whenever $b_{ir} = b_{ci} = 0$ in which case $\Delta_i^\pi = 0$ for all $i \neq r, c$. Otherwise, if $b_{ir} > 0$ and/or $b_{ci} > 0$ the sign of Δ_i^π for $i \neq r, c$ will depend only on α, and is positive (resp. negative) if $\alpha > 0$ (resp. $\alpha < 0$). This completes the proof. □

Proof of Lemma 2. Lemma 1 in Temurshoev (2008) in the framework of social network analysis in mathematically equivalent to Lemma 2 in this paper. Hence, see the proof of Lemma 1 in Temurshoev (2008). □

Derivation of problem (6). As in derivation of problem (4), the objective function in problem (5) can be rewritten as $\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-\{i_1,\ldots,i_k\}} = \boldsymbol{\pi}'(\mathbf{B} - \mathbf{B}^{-\{i_1,\ldots,i_k\}})\mathbf{f} + \boldsymbol{\pi}'\mathbf{B}^{-\{i_1,\ldots,i_k\}}(\mathbf{f} - \mathbf{f}^{-\{i_1,\ldots,i_k\}})$, where $\mathbf{f} - \mathbf{f}^{-\{i_1,\ldots,i_k\}} = \sum_{s=1}^k f_{i_s} \mathbf{e}_{i_s} = \mathbf{E}\mathbf{E}'\mathbf{f}$. This together with Lemma 2 and the fact that $\mathbf{E}\mathbf{E}'\mathbf{E}\mathbf{E}' = \mathbf{E}\mathbf{E}'$ gives

$$\begin{aligned}
&\boldsymbol{\pi}'\mathbf{x} - \boldsymbol{\pi}'\mathbf{x}^{-\{i_1,\ldots,i_k\}} \\
&\quad = \boldsymbol{\pi}' \left[\mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{E}'\mathbf{B} - \mathbf{E}\mathbf{E}'\right] \mathbf{f} + \boldsymbol{\pi}' \left[\mathbf{B} - \mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{E}'\mathbf{B} + \mathbf{E}\mathbf{E}'\right] \mathbf{E}\mathbf{E}'\mathbf{f} \\
&\quad = \boldsymbol{\pi}'\mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{E}'\mathbf{B}\mathbf{f} - \mathbf{E}\mathbf{E}'\mathbf{f} + \boldsymbol{\pi}'\mathbf{B}\mathbf{E}\mathbf{E}'\mathbf{f} - \boldsymbol{\pi}'\mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{E}'\mathbf{B}\mathbf{E}\mathbf{E}'\mathbf{f} + \mathbf{E}\mathbf{E}'\mathbf{f} \\
&\quad = \boldsymbol{\pi}'\mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{E}'\mathbf{B}\mathbf{f} + \boldsymbol{\pi}'\mathbf{B}\mathbf{E}\mathbf{E}'\mathbf{f} - \boldsymbol{\pi}'\mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{B}_{kk}\mathbf{E}'\mathbf{f} = \boldsymbol{\pi}'\mathbf{B}\mathbf{E}\mathbf{B}_{kk}^{-1}\mathbf{E}'\mathbf{B}\mathbf{f},
\end{aligned}$$

Using the fact that for a nonsingular matrix $\mathbf{X}$ the identity $\begin{vmatrix} \mathbf{X} & \mathbf{b} \\ \mathbf{c}' & 0 \end{vmatrix} = -|\mathbf{X}|(\mathbf{c}'\mathbf{X}^{-1}\mathbf{b})$

holds, we can write the hl-th element of the matrix $\mathbf{BEB}_{kk}^{-1}\mathbf{E}'\mathbf{B}$ as follows

$$\mathbf{b}'_{h\bullet}\mathbf{B}_{kk}^{-1}\mathbf{b}_{\bullet l} = \frac{-\begin{vmatrix} \mathbf{B}_{kk} & \mathbf{b}_{\bullet l} \\ \mathbf{b}'_{h\bullet} & 0 \end{vmatrix}}{|\mathbf{B}_{kk}|},$$

where $\mathbf{b}'_{h\bullet}$ is the h-th row of the matrix $\mathbf{BE}$ and $\mathbf{b}_{\bullet l}$ is the l-th column of $\mathbf{E}'\mathbf{B}$. The numerator in the last equation is nothing else as the hl-th element of the matrix field of influence of order k, $\mathbf{F}[(i_1, i_1), (i_2, i_2), \ldots, (i_k, i_k)]$, due to incremental changes in input coefficients $a_{i_1 i_1}, a_{i_2 i_2}, \ldots, a_{i_k i_k}$ (see e.g., Fritz et al., 2002).[40] $\square$

[40]We should note that the only difference comes in signs when k is even, i.e., in the fields of influence approach the determinant in the numerator of the last equation is multiplied by $(-1)^k$. However, we believe that in our setting it should be always multiplied by minus, otherwise the elements will be negative, which then contradict the Leontief inverse property.

Table 4: Codes assigned to 136 Australian sectors

Sym.	Industry	Sym.	Industry
Ac	Insecticides, pesticides and other agricultural chemicals	Lm	Lime
Ai	Aircraft	Lp	Leather and leather products
Al	Aluminium alloys and aluminium recovery	Ma	Agricultural, mining and construction machinery
Ao	Alumina	Mi	Mineral and glass wool and other non-metallic mineral products
Ap	Automotive petrol	Mn	Exploration and services to mining
At	Air and space transport	Mp	Meat and meat products
Ba	Barley, unmilled	Ms	Legal, accounting, marketing and business management services
Bc	Beef cattle	Mv	Motor vehicles and parts, other transport equipment
Bk	Banking	Nb	Non-residential buildings, roads, bridges and other construction
Bl	Black coal	Ne	Newspapers, books, recorded media and other publishing
Bm	Beer and malt	Nf	Non-ferrous metal recovery and basic products
Bp	Bread, cakes, biscuits and other bakery products	Ng	Natural gas
Br	Brown coal, lignite	Oc	Adhesives, inks, polishes and other chemical products
Bs	Typing, copying, staff placement and other business services	Oe	Photographic, optical, medical and radio equipment, watches
Bt	Bus and tramway transport services	Of	Oils and fats
Bu	Prefabricated buildings	Oi	Crude oil
Bv	Soft drinks, cordials and syrups	Om	Coins, jewellery, sporting goods and other manufacturing
Bx	Bauxite	Os	Police, interest groups, fire brigade and other services
Cc	Concrete and mortar	Ot	Cable car, chair lift, monorail and over-snow transport
Ce	Cement	Pa	Paper containers and products
Cg	Services to agriculture, ginned cotton, shearing and hunting	Pc	Petroleum bitumen, refinery LPG and other refinery products
Ch	Basic chemicals	Pd	Property developer, real estate and other property services
Cl	Clothing	Pe	Poultry and eggs
Cm	Communication services	Pg	Pigs
Cn	Confectionery	Ph	Pharmaceutical goods for human use
Co	Copper	Pi	Pipeline transport services
Cp	Plaster and other concrete products	Pl	Plastic products
Cr	Bricks and other ceramic products	Pp	Pulp, paper and paperboard
Cs	Childminding and other community care services	Pr	Printing, stationery and services to printing
Ct	Cosmetics and toiletry preparations	Ps	Hairdressing, goods hiring, laundry and other personal services
Cu	Libraries, parks, museums and the arts	Pt	Paints
Dc	Dairy cattle and untreated whole milk	Rb	Residential building, construction, repair and maintenance
De	Soap and other detergents	Rd	Road freight transport services
Df	Defence	Rf	Railway freight transport services
Dp	Dairy products	Rh	Repairs of household and business equipment
Dw	Ownership of dwellings	Ri	Rice, in the husk
Ed	Education	Rp	Railway passenger transport services
Ee	Cable, wire, batteries, lights and other electrical equipment	Rs	Sport, gambling and recreational services
El	Electricity supply	Rt	Retail trade
En	Electronic equipment, photocopying, gaming machines	Ru	Rubber products
Eq	Pumps, bearings, air conditioning and other equipment	Rv	Repairs of motor vehicles, agricultural and other machinery
Et	Motion picture, radio and television services	Rw	Railway equipment
Fc	Flour, cereal foods, rice, pasta and other flour mill products	Sb	Ships and boats
Fd	Raw sugar, animal feeds, seafoods, coffee and other foods	Sc	Seed cotton
Fe	Mixed fertilisers	Sf	Security broking and dealing and other services to finance
Fi	Commercial fishing	Sg	Sand, gravel and other construction materials mining
Fm	Nuts, bolts, tools and other fabricated metal products	Sh	Sheet containers and other sheet metal products
Fn	Money market corporation and other non-bank finance	Si	Financial asset investors and holding company services
Fo	Gas oil, fuel oil	Sm	Frames, mesh and other structural metal products
Fp	Vegetables, fruit, juices and other fruit and vegetable products	Sp	Water transport
Fr	Forestry and services to forestry	St	Travel agencies, forwarding and other services to transport
Fu	Furniture	Su	Sugar cane
Fw	Footwear	Sw	Softwoods, conifers
Ga	Gas production and distribution	Sz	Silver and zinc ores
Gd	Sanitary and garbage disposal services	Ta	Taxi and hired car with driver
Gl	Gold and lead	Ti	Sawn timer, woodchips and other sawmill products
Gp	Glass and glass products	To	Tobacco products
Gv	Government administration	Tp	Carpets, curtains, tarpaulins, sails, tents and other textiles
Hh	Household appliances and hot water systems	Ts	Scientific research, technical and computer services
Ho	Accommodation, cafes and restaurants	Tx	Processed wool, textile fibres, yarns and woven fabrics
Hs	Health services	Uo	Uranium, nickel, tin, manganese and other non-ferrous metal ores
Hw	Hardwoods, brushwoods, scrubwoods, hewn and other timber	Vf	Vegetable and fruit growing, hay, plant nurseries, flowers
In	Insurance	Wa	Water supply, sewerage and drainage services
Io	Iron ores	Wh	Wheat, legumes for grain, oilseeds, oats and other grains
Is	Basic iron and steel, pipes, tubes, sheets, rods, bars, rails, fittings	Wo	Sheep and shorn wool
Ke	Kerosene and aviation jet fuel	Wp	Plywood, window frames, doors and other wood products
Kn	Knitting mill products	Ws	Wine and spirits
Lg	Liquefied natural gas, liquefied natural petrol	Wt	Wholesale trade

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